

Assignment-
Sub: Applied Mathematics

Sel-1

Year: 2024
Sem.: Spring.

1. a] Define analyticity of a function $f(z)$. Show that necessary condition for the function $f(z) = u(x, y) + i v(x, y)$ to be analytic on a domain D is $u_x = v_y$ and $u_y = -v_x$ at each point (x, y) of D .

$\Rightarrow$ A function $f(z)$ is said to be analytic in a domain D if and only if the first derivatives of u and v satisfies the two. and also so called Cauchy Riemann's eqⁿ.

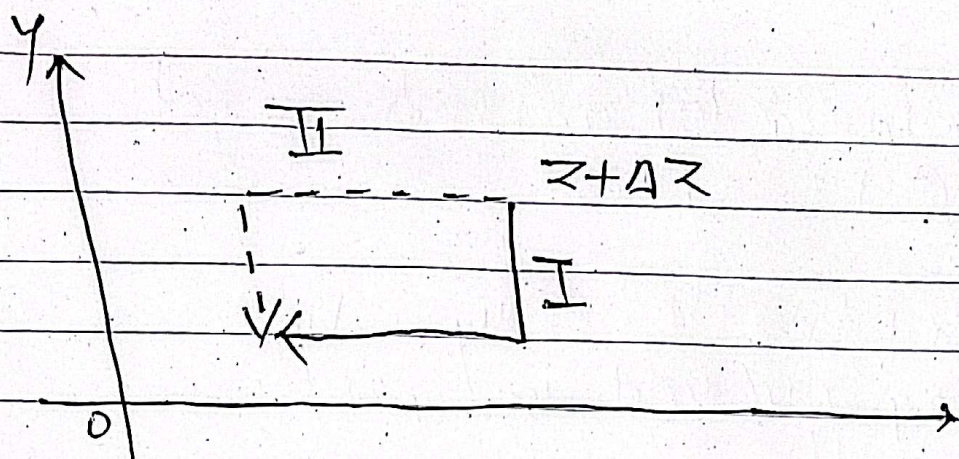
$$u_x = v_y \text{ and } u_y = -v_x \text{ every where in } D.$$

We have $f(z) = u(x, y) + i v(x, y)$ differentiable then that means $f'(z)$ exists.

Where,

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{[u(x + \Delta x, y + \Delta y) + i v(x + \Delta x, y + \Delta y)] - [u(x, y) + i v(x, y)]}{(\Delta x + i \Delta y)} \quad \text{--- (i)}$$



We have two paths to move on a plane starting from x-axis and y-axis as you choose.

In eqn (i): First we choose path - I where $\Delta y \rightarrow 0$ and $\Delta x \rightarrow 0$. After Δy is zero, $\Delta z = \Delta x$ the eqn (i) becomes.

$$f'(z) = \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + \lim_{\Delta x \rightarrow 0} i \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}$$

$$= u_x + i v_x \quad \text{--- (ii)}$$

Again, we choose path - II, where $\Delta x \rightarrow 0$ in eqn (i). After Δx is zero, $\Delta z = i \Delta y$ then eqn (i) becomes.

$$f'(z) = \lim_{\Delta y \rightarrow 0} \frac{u(x, y+\Delta y) - u(x, y)}{i \Delta y} + \lim_{\Delta y \rightarrow 0} i \frac{v(x, y+\Delta y) - v(x, y)}{i \Delta y}$$

$$= \frac{1}{i} u_y + v_y = -i u_y + v_y \quad \text{--- (iii)}$$

from eqn (ii) and (iii) we get,
 $f'(z) = f'(z)$

$$u_x + i v_x = -i u_y + v_y$$

Equating real and im. part,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{Hence proved.}$$

1. b] State and prove Cauchy integral formula.
Evaluate the integral $\oint_C \frac{e^{sz}}{(z+i)^4} dz$ Where

$$C: |z| = 2$$

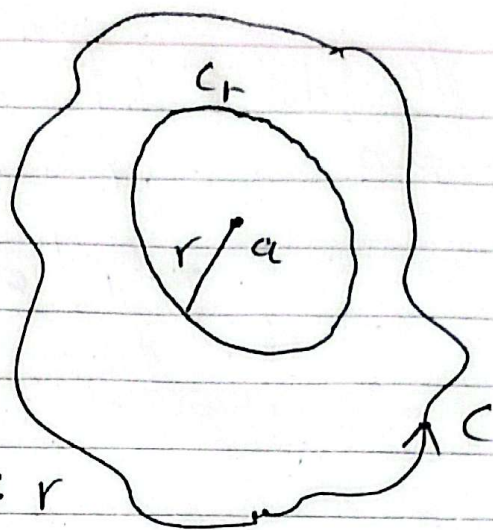
$\Rightarrow$ Statement: If $f(z)$ is analytic inside and on a curve C of simply connected region R and if 'a' is any point in C , then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)} dz, \text{ integration being taken Counterclockwise.}$$

$$\therefore \oint_C \frac{f(z)}{(z-a)} = 2\pi i f(a).$$

Proof: We have $f(z)$ is analytic inside and on C .

$\frac{f(z)}{z-a}$ is analytic inside and on C except at a .
Then draw a small circle C_r with centre a and radius r as shown in figure.



Here, $\frac{f(z)}{z-a}$ is analytic in the multiply connected region bounded by C and C_r .

then

Cauchy integral theorem,

$$\oint_C \frac{f(z)}{z-a} dz = \oint_{C_r} \frac{f(z)}{z-a} dz \quad \text{--- (i)}$$

on C_r : $z = a + re^{i\theta}$

$$dz = ri e^{i\theta} d\theta$$

$$\begin{aligned} \text{Thus, } \oint_C \frac{f(z)}{z-a} dz &= \oint_{C_r} \frac{f(z)}{z-a} dz = \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}} \cdot ri e^{i\theta} d\theta \\ &= i \int_0^{2\pi} f(a + re^{i\theta}) d\theta \end{aligned}$$

As $r \rightarrow 0$, then

$$\oint_C \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a) d\theta$$

$$= i f(a) 2\pi$$

$$= 2\pi i f(a)$$

hence proved.

here, Given, $\oint \frac{e^{sz}}{(z+i)^4} dz$ $C: |z|=2$

$$f(z) = e^{sz}$$

Test analyticity.

$$(z+i)^4 = 0 \Rightarrow z = -i$$

The pole of order 4. $\rightarrow n+r = 3+r$ then
 $n=3$.

By Cauchy integral formula,

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz. \quad \text{--- (1)}$$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

$$\oint_C \frac{f(z)}{(z-z_0)^4} dz = \frac{2\pi i}{3!} f'''(z_0) \quad \text{--- (4)}$$

Here, $f(z) = e^{5z}$

$$f'(z) = 5e^{5z}$$

$$f''(z) = 25e^{5z}$$

$$f'''(z) = 125e^{5z}$$

$f'''(-i) = 125e^{-5i}$ then,
eqn (4) becomes,

$$\oint_C \frac{e^{5z}}{(z+i)^4} dz = \frac{2\pi i}{3!} 125e^{-5i}$$

$$= \frac{125\pi i}{3} e^{-5i} \quad \#$$

2a] Find the expansion of $\frac{z-2}{z(z+1)(z-2)}$ in the

region given by

i) $0 < |z+1| < 1$.

ii) $1 < |z+1| < 3$.

$\Rightarrow$ Here, Given $\frac{z-2}{z(z+1)(z-2)}$

funcⁿ is not analytic at $z(z+1)(z-2) = 0$
 $\Rightarrow z = 0, -1, 2$

By partial fraction,

$$\frac{z-2}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{(z+1)} + \frac{C}{(z-2)} \quad \text{--- (1)}$$

$$z-2 = A(z+1)(z-2) + B(z)(z-2) + C(z)(z+1)$$

At $z=0$, then, $-2 = A(-2) \Rightarrow A = 1$.

At $z=-1$, then,

$$-9 = B(-1) \cdot (-3) \Rightarrow B = -3$$

At $z=2$ then, $12 = C(2) \cdot (3) \Rightarrow C = 2$

eq (i) becomes,

$$f(z) = \frac{z-2}{z(z+1)(z-2)} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{(z-2)} \quad \text{--- (ii)}$$

By Maclaurin's series expansion,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n \text{ for } |z| < 1.$$

∴ $0 < |z+1| < 1 \Rightarrow 0 < |z+1| \text{ and } |z+1| < 1$

$$f(z) = \frac{1}{z+1-1} = \frac{3}{z+1} + \frac{2}{z+1-3} \quad \text{[from (ii)]}$$

$$= -\frac{3}{z+1} = \frac{1}{[1-(z+1)]} + \frac{2}{3} \left[\frac{1}{1 - \left(\frac{z+1}{3}\right)} \right]$$

$$= -\frac{3}{(z+1)} = \frac{1}{1+(z+1)} + \frac{2}{3} \left[\frac{1}{1 - \left(\frac{z+1}{3}\right)} + \frac{2}{3} \left[1 + \frac{z+1}{3} + \right. \right.$$

$$\left. \left(\frac{z+1}{3} \right)^2 + \left(\frac{z+1}{3} \right)^3 + \dots \right]$$

$$= -\frac{3}{(z+1)} = \sum_{n=0}^{\infty} (z+1)^n + \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{z+1}{3} \right)^n$$

$$i.i) 1 < |z+1| < 3 \Rightarrow \frac{1}{|z+1|} < 1 \text{ and } \left| \frac{z+1}{3} \right| < 1$$

$$f(z) = \frac{1}{\frac{1}{z+1}} - \frac{3}{z+1} + \frac{2}{(-3)} \left(\frac{1}{1 - \left(\frac{z+1}{3} \right)} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(z+1)^n} - \frac{3}{z+1} - \frac{2}{3} \left(\sum_{n=0}^{\infty} \left(\frac{z+1}{3} \right)^n \right)$$

||

2.6] Define bilinear transformation. Find the bilinear transformation which maps the points $z=0, -1, i$ onto the points $w=i, 0, \infty$. Also, find the image of the unit circle $|z|=1$.

$\Rightarrow$ A bilinear transformation is a function of the form,

$$w = \frac{az+b}{cz+d}$$

Where a, b, c, d are complex constant such that $ad-bc \neq 0$.

Given that

$$z = 0, -1, i$$

$$w = i, 0, \infty$$

We know that, bilinear transformation of z, z_1, z_2, z_3 in z -plane and w, w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \text{--- (i)}$$

Substituting the given values in eqⁿ (i),

$$\frac{(w-i)(0-\infty)}{(w-\infty)(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \lim_{r \rightarrow \infty} \frac{(w-i)r}{(w-r)i} = \frac{z(1+i)}{(z-i)}$$

$$\Rightarrow \frac{z(1+i)}{(z-i)} = \lim_{r \rightarrow \infty} \frac{(w-i)r}{(w-r)i} \quad \left[\text{in } \frac{\infty}{\infty} \text{ form} \right]$$

$$= \lim_{r \rightarrow \infty} \frac{(w-i)}{-i}$$

$$= \frac{(w-i)}{(-i)} = iw + 1$$

$$\Rightarrow \frac{z(1+i)}{(z-i)} = i\omega + 1$$

$$\Rightarrow \omega = \frac{1}{i} \left[\frac{z(1+i)}{(z-i)} - 1 \right]$$

$$= \frac{1}{i} \left[\frac{z(1+i) - z + i}{(z-i)} \right]$$

$$= \frac{1}{i} \left[\frac{zi + i}{(z-i)} \right] = \frac{z+i}{z-i}$$

This is required bilinear transformation

$$\text{here. } \omega = \frac{z+i}{z-i} \Rightarrow zw - i\omega = z + i$$

$$\Rightarrow zw - z = 1 + i\omega$$

$$\Rightarrow z(\omega - 1) = 1 + i\omega$$

$$\Rightarrow z = \frac{1 + i\omega}{(\omega - 1)}$$

Again.

Given, region is $|z| = r$ so,

$$\frac{1 + i\omega}{\omega - 1} = 1 \Rightarrow |i(-i + \omega)| = |\omega - 1|$$

$$\Rightarrow (-i + \omega)(-i + \bar{\omega}) = (\omega - 1)(\bar{\omega} - 1)$$

$$\Rightarrow -L - i(\bar{w} + w) + w\bar{w} = w\bar{w} - w - \bar{w} + L$$

$$\Rightarrow -2iY = -2Y + 2$$

$$\Rightarrow -iY = -Y + 1$$

$$\Rightarrow \dots \dots \dots iY = Y - 1$$

This means the image of $|z|=L$ is $V=Y-1$. #

3.a] State and prove first Shifting theorem on z-transform. find z-transform of $e^{in\pi/2}$ and find $\mathcal{Z}(\cos \frac{n\pi}{2})$ and $\mathcal{Z}(\sin \frac{n\pi}{2})$.

$\Rightarrow$ Statement: If $\mathcal{Z}[f(t)] = F(z)$, then $\mathcal{Z}[e^{-at} f(t)] = F(e^{at} z)$.

Proof: We have by defⁿ.

$$\mathcal{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad \text{--- (i)}$$

$$\begin{aligned} \Rightarrow \mathcal{Z}[e^{-at} f(t)] &= \sum_{n=0}^{\infty} [e^{-anT} f(nT) z^{-n}] \\ &= \mathcal{Z}[e^{at} z] \text{ proved.} \end{aligned}$$

⇒ Here. We know,

$$Z(a^n) = \frac{Z}{Z-a}$$

$$\text{Thus. } Z(e^{in\pi/2}) = Z[(e^{i\pi/2})^n]$$

$$= \frac{Z}{Z - e^{i\pi/2}}$$

$$= \frac{Z}{Z - (\cos \pi/2 + i \sin \pi/2)}$$

$$\Rightarrow Z\left(\frac{\cos n\pi}{2} + i \frac{\sin n\pi}{2}\right) = \frac{Z}{(Z-i)}$$

$$= \frac{Z(Z+i)}{Z^2+1}$$

$$\Rightarrow Z\left(\frac{\cos n\pi}{2}\right) + i Z\left(\frac{\sin n\pi}{2}\right) = \frac{Z^2}{Z^2+1} + i \frac{Z}{Z^2+1}$$

Equating,

$$Z\left(\frac{\cos n\pi}{2}\right) = \frac{Z^2}{Z^2+1}$$

$$Z\left(\frac{\sin n\pi}{2}\right) = \frac{Z}{Z^2+1}$$

#

3.b) Solve the difference equation $y_{n+2} - 7y_{n+1} + 12y_n = 2n$, $y_0 = 0$, $y_1 = 0$ by using z-transform

⇒ Given, $y_{n+2} - 7y_{n+1} + 12y_n = 2n$, $y_0 = y_1 = 0$

Taking z-transform on both sides

$$Z(y_{n+2}) - 7Z(y_{n+1}) + 12Z(y_n) = 2Z(n)$$

$$\Rightarrow z^2(\bar{y} - y_0 - \frac{y_1}{z}) - 7z(\bar{y} - y_0) + 12\bar{y} = \frac{2z}{(z-1)^2}$$

$$\Rightarrow z^2\bar{y} - 7z\bar{y} + 12\bar{y} = \frac{2z}{(z-1)^2}$$

$$\bar{y}(z^2 - 7z + 12) = \frac{2z}{(z-1)^2}$$

$$\bar{y} = \frac{2z}{(z-1)^2(z^2 - 7z + 12)}$$

$$\bar{y} = \frac{2z}{(z-1)^2(z-3)(z-4)} \quad \text{--- (i)}$$

~~$$\frac{2z}{(z-1)^2(z-3)(z-4)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-3} + \frac{D}{z-4}$$~~

$$2z = \cancel{A(z-1)(z-3)(z-4)} + \cancel{B(z-3)(z-4)} + \cancel{C(z-1)^2(z-4)} + D(z-1)^2(z-3)$$

$$\cancel{A(z-1)}, \quad B(-2) \cdot (-3) = 2 \Rightarrow B = \frac{2}{6} = \frac{1}{3}$$

$$\cancel{A(z-3)}, \quad 3C(2)^2(1) = 6 \Rightarrow C = \frac{6}{12} = \frac{1}{2}$$

The poles of $(z-1)^2(z-3)(z-4) = 0$
 $\Rightarrow z = 1, 3, 4$

Res. at $z=1$ simple pole of order 2. then,

$$\text{Res.}[f(z), 1] = \lim_{z \rightarrow 1} \frac{d}{dz} \left[\frac{2z}{(z-1)^2(z-3)(z-4)} \right]$$

$$= \lim_{z \rightarrow 1} \left[\frac{2n z^{n-1} (z^2 - 7z + 12) - 2z^n (2z - 7)}{(z^2 - 7z + 12)^2} \right]$$

$$= \frac{6n + 5}{18}$$

Res. at $z=3$ simple pole.

$$\text{Res.}[f(z), 3] = \lim_{z \rightarrow 3} \left[\frac{2z}{(z-1)(z-3)(z-4)} \right] = \frac{-3^n}{2}$$

Res. at $z=4$ simple pole. then.

$$\text{Res. } [f(z), 4] = \lim_{z \rightarrow 4} \left[\cancel{(z-4)} \cdot \frac{2z}{(z-1)^2(z-3)\cancel{(z-4)}} \right]$$
$$= \frac{2 \cdot 4^n}{9}$$

Now, for $f(n) = \text{sum of all residues.}$

$$= \frac{6n+5}{18} + \left(-\frac{3^n}{2} \right) + \frac{2 \cdot 4^n}{9}$$

$$= \frac{6n+5}{18} - \frac{3^n}{2} + \frac{2 \cdot 4^n}{9} \quad \#$$

4.a] Show that $z[n f(t)] = -z \frac{d}{dz} [F(z)]$ where $f(z) = z[f(t)]$. find $z^{-1} \left[\frac{z}{(z+1)^2(z-1)} \right]$.

$\Rightarrow$ By definition of z-transform.

$$z[n f(t)] = \sum_{n=0}^{\infty} [n f(nT)] z^{-n} \quad \text{--- (i)}$$

$$F(z) = z[f(t)]$$

$$= \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$\frac{d}{dz} [F(z)] = \sum_{n=0}^{\infty} -n z^{-(n+1)} f(nT) \quad \text{--- (ii)}$$

$$= -z^{-1} \sum_{n=0}^{\infty} n z^{-n} f(nT)$$

Using eqⁿ (i) and (ii), we get.

$$z[n f(t)] = -z \frac{d}{dz} F(z)$$

proved.

Given. $z^{-1} \left[\frac{z}{(z+1)^2(z-1)} \right]$

let $\frac{z}{(z+1)^2(z-1)} = \frac{Az}{(z+1)} + \frac{Bz}{(z+1)^2} + \frac{Cz}{z-1}$

$$z = Az(z+1)(z-1) + Bz(z-1) + Cz(z+1)^2$$

At $z=1$, $C(4) = 1 \Rightarrow C = 1/4$

At $z=-1$, $-1 = 2B \Rightarrow B = -1/2$

Equating. ~~z~~³ then,

$$0 = A + C \Rightarrow A = -C = -1/4$$

$$z^{-1} \left(\frac{z}{(z+1)^2(z-1)} \right) = \frac{-1}{4} z^{-1} \left(\frac{z}{z+1} \right) - \frac{1}{2} z^{-1} \left(\frac{z}{(z+1)^2} \right) + \frac{1}{4} z^{-1} \left(\frac{z}{z-1} \right)$$

$$= \frac{-1}{4} (-1)^n - \frac{1}{2} n(-1)^n + \frac{1}{4} (1)^n$$

#

4.6] Find the solution of one dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ with initial velocity $g(x)$ initial deflection $f(x)$ and boundary condition $y(0, t) = 0 = y(L, t)$.

$\Rightarrow$ The 2 one dim. wave eqn is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (i)}$$

Where, $y(x, t)$ is the deflection of string with boundary condⁿ.

$$y(0, t) = 0 \text{ and } y(L, t) = 0 \text{ for all } t \quad \text{--- (ii)}$$

And, initial condⁿ,

$$y(x, 0) = f(x) = \text{initial deflection} \quad \text{--- (iii)}$$

$$\left. \frac{\partial y}{\partial t} \right|_{t=0} = g(x) = \text{initial velocity} \quad \text{--- (iv)}$$

$$\text{Let } u(x, t) = f(x) \cdot G(t) \quad \text{--- (v)}$$

Diff. eqn (v) partially w.r. to t and x , we get,

$$\frac{\partial^2 u}{\partial x^2} = f'' G \Rightarrow \frac{\partial^2 u}{\partial x^2} = f'' G$$

$$\frac{\partial^2 u}{\partial t^2} = f \ddot{G} \Rightarrow \frac{\partial^2 u}{\partial t^2} = f \ddot{G}$$

then eqn (i) becomes, $fG'' = c^2 F'' G$

$$\Rightarrow \frac{G''}{c^2 G} = \frac{F''}{F} = k \text{ say, This gives.}$$

$$G'' - c^2 k G = 0 \quad \text{--- (vi)}$$

$$F'' - Fk = 0 \quad \text{--- (vii) which are ordinary diff. eqn.}$$

Now, we have to find the value of f and G that depends on the eqn (vi) and (vii) with boundary condⁿ.

After solving (vi) we get, $f = Ae^{kx} + Be^{-kx}$
where k is positive. let $k = \lambda^2$ then,

$$f = Ae^{\lambda x} + Be^{-\lambda x}$$

If taking k is (-ve) as $k = -p^2$ then,

$$f'' + p^2 f = 0 \quad \text{[by (vi)]}$$

$$m^2 + p^2 = 0$$

$$m = \pm pi$$

$$f(x) = A \cos px + B \sin px$$

We have, $f(0) = A = 0$

$$f(L) = B \sin pL = 0$$

$$\sin pL = 0 \quad \text{Where } B \neq 0$$

$$pL = n\pi$$

$$p = \frac{n\pi}{L} \quad \text{for } n \text{ is a integer.}$$

We have $B \neq 0$, setting $B = 1$;

$$f(x) = \sin \frac{n\pi}{L} x, \quad \text{we obtain infinitely many sol}^n.$$

$$f(x) = f_n(x). \quad \text{thus,}$$

$$f_n(x) = \sin \frac{n\pi}{L} x \quad \text{for } n = 1, 2, 3, \dots \quad \text{--- (4)}$$

$$\text{Solving eqn (vii), } k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$$

$$\ddot{G} - c^2 k G = 0$$

$$\Rightarrow \ddot{G} + \lambda_n^2 G = 0 \quad \text{Where } \lambda_n = \frac{c n \pi}{L}$$

$$m^2 + \lambda_n^2 = 0 \Rightarrow m = \pm i \lambda_n$$

It's solⁿ,

$$G_n(x) = B_n \cos \lambda_n x + B_n^* \sin \lambda_n x$$

$\therefore$ The required solution of eqn (i) is.

$$y(x, t) = f(x) \cdot G(t)$$

$$u_n(x, t) = \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t \right) \sin \frac{n\pi}{L} x$$

————— (A)

Where, The set of $(\lambda_1, \lambda_2, \lambda_3, \dots)$ is called the spectrum of the vibrating string and its value is.

$$\lambda_n = \frac{n\pi}{L}$$

5.a] Express the Laplacian $\nabla^2 y = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$ in Polar Coordinate.

$\Rightarrow$ We have, Laplacian in Cartesian coordinate is

$$\nabla^2 y = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \quad \text{————— (i)}$$

We have to change in polar coordinate
Put $x = r \cos \theta$ and $y = r \sin \theta$

then, $y_x = \frac{\partial y}{\partial x} = y_r r_x + y_\theta \theta_x$

Again,

diff. with r to x .

$$u_{xx} = (u_r r_x)_x + (u_\theta \theta_x)_x$$

$$= (u_r)_x r_x + u_r r_{xx} + u_{\theta x} \theta_x + u_\theta \theta_{xx}$$

$$= (u_{rr} r_x + u_r \theta \theta_x) r_x + u_r r_{xx} x$$

$$+ (u_{\theta r} r_x + u_{\theta \theta} \theta_x) \theta_x + u_\theta \theta_{xx} \quad \text{--- (ii)}$$

We have. $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} y/x$

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}$$

$$\text{and } \theta_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2}\right) = \frac{-y}{r^2}$$

$$\text{and } r_{xx} = \frac{r - x r_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3}$$

$$\text{and } \theta_{xx} = -y \left(\frac{-2^2}{r^3} \right) \frac{1}{x} = \frac{2xy}{r^4}$$

Putting these value in eqn (i),

$$y_{xx} = \frac{x^2}{r^2} y_{rr} - \frac{2xy}{r^3} y_{r\theta} + \frac{y^2}{r^4} y_{\theta\theta} + \frac{y^3}{r^3} y_r + \frac{2xy}{r^4} y_\theta$$

— (iii)

Similarly,

$$y_{yy} = \frac{y^2}{r^2} y_{rr} + \frac{2xy}{r^3} y_{r\theta} + \frac{x^2}{r^4} y_{\theta\theta} + \frac{x^2}{r^3} y_r - \frac{2xy}{r^4} y_\theta$$

— (iv)

from eqn. (iii) and (iv).

$$\nabla^2 y = \frac{\partial^2 y}{\partial r^2} + \frac{1}{r} \frac{\partial y}{\partial r} + \frac{1}{r^2} \frac{\partial^2 y}{\partial \theta^2}$$

This is said to be Laplacian of y in polar coordinate.

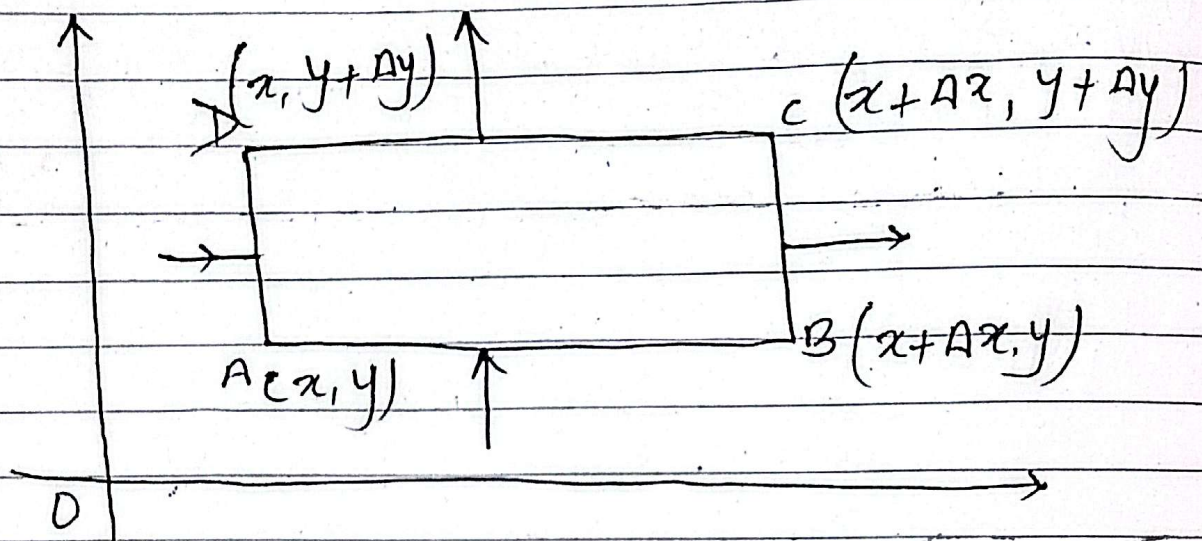
Similarly,

The Laplacian of y in cylindrical co-ordinate.

$$\nabla^2 y = y_{rr} + \frac{1}{r} y_r + \frac{1}{r^2} y_{\theta\theta} + y_{zz}$$

Q. Derive two dimensional heat equation completely with necessary assumptions.

⇒ let us suppose that the flow of heat in a metal plate in XY plane.



Take a rectangular element of the plate with side Δx and Δy and thickness τ . then.

the side AB and AD is given by

$$-k\tau\Delta x\left(\frac{\partial y}{\partial y}\right)_x \text{ and } -k\tau\Delta y\left(\frac{\partial y}{\partial x}\right)_x$$

respectively.

and sides CD and BC per second is,

$$-k\tau\Delta x\left(\frac{\partial y}{\partial y}\right)_{y+\Delta y} \text{ and } -k\tau\Delta y\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} \text{ respectively.}$$

$\therefore$ The total gain of heat by the rectangular plate ABCD per second is.

$$= -k\alpha \Delta x \left(\frac{\partial y}{\partial x} \right)_y - k\alpha \Delta y \left(\frac{\partial y}{\partial x} \right)_x + k\alpha \Delta x \left(\frac{\partial y}{\partial x} \right)_{y+\Delta y} + k\alpha \Delta y \left(\frac{\partial y}{\partial x} \right)_{x+\Delta x}$$

$$= k\alpha \Delta x \Delta y \left[\frac{\left(\frac{\partial y}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x} \right)_x}{\Delta x} + \frac{\left(\frac{\partial y}{\partial x} \right)_{y+\Delta y} - \left(\frac{\partial y}{\partial x} \right)_y}{\Delta y} \right]$$

Also, the rate of gain of heat by plate is. — (i)

$$S \rho \Delta x \Delta y \frac{\partial y}{\partial t} \quad \text{--- (ii)}$$

Where S = specific heat

ρ = density of metal plate.

From (i) and (ii),

$$k\alpha \Delta x \Delta y \left[\frac{\left(\frac{\partial y}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x} \right)_x}{\Delta x} + \frac{\left(\frac{\partial y}{\partial x} \right)_{y+\Delta y} - \left(\frac{\partial y}{\partial x} \right)_y}{\Delta y} \right] = S \rho \Delta x \Delta y \frac{\partial y}{\partial t}$$

Dividing both sides by $\alpha \Delta x \Delta y$ and taking limit as $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$ on both sides,

$$k \left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \right) = \rho s \frac{\partial y}{\partial t}$$

$$\Rightarrow \frac{\partial y}{\partial t} = \frac{k}{\rho s} \left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \right)$$

$$= c^2 \left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \right) \quad \text{--- (A)}$$

Where $c^2 = k/\rho s$

Which is required two dimensional heat eqⁿ.

5.6] Find the temp^r in the laterally insulated bar of length L whose ends are kept at temp^r 0, assuming that the initial temp^r is

$$f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L-x & \text{if } L/2 < x < L \end{cases}$$

$\Rightarrow$ Given,

length = L

initial deflection $y(x, 0) = f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L-x & \text{if } L/2 < x < L \end{cases}$

And $y(0, t) = 0 = y(L, t)$

Also, $\lambda_n = \frac{cn\pi}{L}$

We know, the solⁿ of one dimen. heat eqⁿ is.

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x e^{-\lambda_n^2 t} \quad \text{--- (i)}$$

Where,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx.$$

Here,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \left(\frac{n\pi}{L} x \right) x \cdot dx.$$

$$= \frac{2}{L} \left[\int_0^{L/2} x \sin \frac{n\pi}{L} x dx + \int_{L/2}^L (L-x) \sin \frac{n\pi}{L} x dx \right]$$

$$= \frac{2}{L} \left[\left(-\frac{x \cos \frac{n\pi}{L} x}{\left(\frac{n\pi}{L} \right)} + \frac{\sin \frac{n\pi}{L} x}{\left(\frac{n\pi}{L} \right)^2} \right) \Big|_0^{L/2} + \left(\frac{(L-x) \cos \frac{n\pi}{L} x}{\left(-\frac{n\pi}{L} \right)} - \frac{\sin \frac{n\pi}{L} x}{\left(\frac{n\pi}{L} \right)^2} \right) \Big|_{L/2}^L \right]$$

$$= \frac{2}{L} \left\{ -\frac{L^2}{2n\pi} \cos \left(\frac{n\pi}{2} \right) + \frac{L^2}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right) + \frac{L^2}{2n\pi} \cos \left(\frac{n\pi}{2} \right) + \frac{L^2}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right) \right\}$$

$$= \frac{4L}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right)$$

Therefore (i) becomes:

$$u(x, t) = \sum_{n=1}^{\infty} \frac{4L}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) \cdot \sin\left(\frac{n\pi}{L}\right) x \cdot e^{-\lambda_n t}$$

#

6.a] Find the fourier cosine transform of $f(x) = e^{-mx}$ for $m > 0$, and then show that

$$\int_0^{\infty} \frac{\cos kx}{1+x^2} dx = \frac{\pi}{2} e^{-k}$$

$\Rightarrow$ let $f(x) = e^{-mx}$ for $m > 0$

then the fourier cosine transform of $f(x)$ is

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-mx} \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-m\omega}}{m^2 + \omega^2} (-m \cos \omega x + \omega \sin \omega x) d\omega$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{m}{m^2 + \omega^2} \right) \quad \text{--- } \odot$$

Now, we have to show,

$$\int_0^{\infty} \frac{\cos kx}{(1+x^2)} dx = \frac{\pi}{2} e^{-k} \quad \text{then,}$$

We have the inverse fourier cosines transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(x) \cos \omega x d\omega$$

$$e^{-mx} = \frac{2}{\pi} \int_0^{\infty} \frac{m \cos \omega x}{m^2 + \omega^2} d\omega \quad [\text{using (i)}]$$

Set $x=k$, $m=k > 0$, then $\omega=x$ so,

$$e^{-k} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos xk}{1+x^2} dx$$

$$\therefore \int_0^{\infty} \frac{\cos xk}{1+x^2} dx = \frac{\pi}{2} e^{-k} \quad \text{proved.}$$

6.6] Show that $\int_0^{\infty} \frac{\cos \omega x + \omega \sin \omega x}{1 + \omega^2} d\omega = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$

$\Rightarrow$ Given, $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$ — (i)

to show $f(x) = \int_0^{\infty} \left(\frac{\cos \omega x + \omega \sin \omega x}{1 + \omega^2} \right) d\omega$ — (ii)

the fourier integral for $f(x)$ is,

$f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$ — (iii)

where,

$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t dt$

$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t dt$

Here,

$A(\omega) = \frac{1}{\pi} \left[\int_{-\infty}^0 0 \cdot \cos \omega t dt + \int_0^{\infty} \pi e^{-t} \cos \omega t dt \right]$

$$= \frac{\pi}{\pi} \int_0^{\infty} e^{-t} \cos \omega t \, dt$$

$$= \left[\frac{e^{-t}}{1+\omega^2} (-1 \cos \omega t + \omega \sin \omega t) \right]_0^{\infty}$$

$$= \frac{1}{1+\omega^2} [\because e^{-\infty} = 0]$$

And, $B(\omega) = \frac{1}{\pi} \int_0^{\infty} e^{-t} \sin \omega t \, dt$

$$= \left[\frac{e^{-t}}{1+\omega^2} (-\sin \omega t + \omega \cos \omega t) \right]_0^{\infty}$$

$$= \frac{\omega}{1+\omega^2}$$

then eqn (iii) becomes

$$f(x) = \int_0^{\infty} \frac{(\cos \omega x + \omega \sin \omega x)}{1+\omega^2} d\omega$$

#

hence proved

7.a] check analyticity of $f(z) = z^2$

$$\Rightarrow f(z) = z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 \\ = x^2 - y^2 + i2xy$$

Comparing, $U + iV$ then,

$$U = x^2 - y^2, \quad V = 2xy$$

By C-R eqn.

$$U_x = 2x$$

$$U_y = -2y$$

$$V_x = 2y$$

$$V_y = 2x$$

then,

$U_x = V_y$ and $U_y = -V_x$ then
the given function is analytic.

7(b) Find $\sum (a^n)$

$$\Rightarrow \text{Here } \sum (a^n) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left[\left(\frac{a}{z} \right)^n \right] \\ = 1 + \frac{a}{z} + \left(\frac{a}{z} \right)^2 + \dots \\ = \left(1 - \frac{a}{z} \right)^{-1}$$

$$f(a) = \frac{z}{z-a}$$

1/2

7.C.] Find the solution of partial differential eqn
 $4x^2 + 9y = 0$

$$\Rightarrow \text{Here, } 4x^2 + 9y = 0$$

$$m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

then it's soln is.

$$y = A \cos 3x + B \sin 3x \quad \text{Where A and B are constant.}$$

7.D.] Define linear partial differential eqn with suitable example.

$\Rightarrow$ A eqn involving one or more partial derivatives of an (unknown) funcn of two or more independent variable's called linear partial differential eqn.

For Example: one dimen. wave eqn: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

one. Dim. heat eqn: $\frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2}$

Set-2

Year: 2023
Sem: Spring

1.a] Define the harmonic function, Is a function harmonic? If yes, find a corresponding harmonic conjugate and the analytic function.

$$V = 2xy - \frac{y}{x^2 + y^2}$$

⇒ Harmonic function: A real funcⁿ of two variable x and y that posses continuous second order partial derivatives and that satisfies Laplace eqⁿ is called harmonic function.

Here, $V = 2xy - \frac{y}{x^2 + y^2}$

Diff. we get,

$$\begin{aligned} V_x &= 2y + y(x^2 + y^2)^{-2} \cdot 2x \\ &= 2y + \frac{2xy}{(x^2 + y^2)^2} \end{aligned}$$

Again,

$$\begin{aligned}
 V_y &= 2x - [4(-1)(x^2+y^2)^{-3} 2y + (x^2+y^2)^{-1} \cdot 1] \\
 &= 2x + \frac{2y^2}{(x^2+y^2)^3} - \frac{1}{(x^2+y^2)} \\
 &= 2x + \frac{2y^2 - (x^2+y^2)}{(x^2+y^2)^2} \\
 &= 2x - \frac{x^2 - y^2}{(x^2+y^2)^2}
 \end{aligned}$$

$$\begin{aligned}
 V_{xx} &= 2xy(x^2+y^2)^{-3}(-2) \cdot 2x + (x^2+y^2)^{-2} 2y \\
 &= -\frac{8x^2y}{(x^2+y^2)^3} + \frac{2y}{(x^2+y^2)^2} \\
 &= \frac{-8x^2y + 2x^2y + 2y^3}{(x^2+y^2)^3} \\
 &= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}
 \end{aligned}$$

$$\begin{aligned}
 v_{yy} &= 0 - \frac{(x^2 - y^2)(-2)(x^2 + y^2)^{-3} \cdot 2y - (x^2 + y^2)^{-2}(-2y)}{4y(x^2 - y^2)} + \frac{2y}{(x^2 + y^2)^2} \\
 &= \frac{4yx^2 - 4y^3 + 2yx^2 + 2y^3}{(x^2 + y^2)^3} \\
 &= \frac{6x^2y - 2y^3}{(x^2 + y^2)^3}
 \end{aligned}$$

Now, $v_{xx} + v_{yy} = 0$ Thus, by definition v is harmonic function.

Let $f(z) = u + iv$ be analytic. By c.f eqⁿ.

$$u_y = -v_x = -2y + \frac{2xy}{(x^2 + y^2)^2}$$

Integrating w.r. to y ,

$$u = -y^2 + \frac{x}{x^2 + y^2} + h(x) \quad \text{--- (i)}$$

$$\begin{aligned}
 \text{Then, } u_x &= x(-1)(x^2 + y^2)^{-2} \cdot 2x + (x^2 + y^2)^{-1} \cdot h'(x) \\
 &= \frac{-2x^2 + x^2 + y^2}{(x^2 + y^2)^2} + h'(x)
 \end{aligned}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2} + h'(x)$$

since, $u_x = v_y$ so, $h'(x) = 2x$

integrate, $h(x) = x^2 + c$ so, eqn (4) becomes.

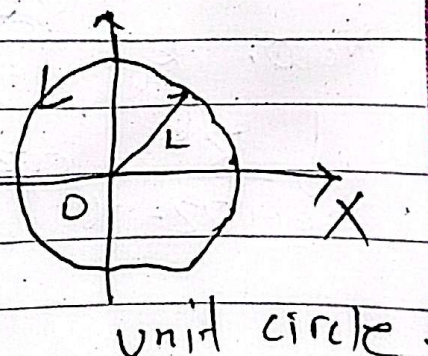
$$u = -y^2 + \frac{x}{x^2 + y^2} + x^2 + c$$

$$= x^2 - y^2 + \frac{x}{x^2 + y^2} + c \quad \text{Which is required conjugate.}$$

1.6] state Cauchy integral formula for derivative.
Evaluate $\oint_C \frac{z^6}{(2z-1)^6} dz$ where C is unit circle $|z| = 1$ counterclockwise.

$\Rightarrow$ we have, $\oint_C \frac{z^6}{(2z-1)^6} dz$.

here, $f(z) = \frac{z^6}{(2z-1)^6}$ which does not exist,



$$\text{at } 2z-1=0 \\ \Rightarrow z = 1/2 \text{ which lies within } C.$$

By Cauchy integral formula.

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0).$$

$$\text{i.e. } \oint \frac{z^6}{(2z-1)^{5+1}} dz = \frac{2\pi i}{2^6 5!} f^{(5)}(1/2) \quad \text{--- (1)}$$

$$\text{Here, } f(z) = z^6$$

$$f'(z) = 6z^5$$

$$f''(z) = 30z^4$$

$$f'''(z) = 120z^3$$

$$f^{(4)}(z) = 360z^2$$

$$f^{(5)}(z) = 720z$$

$$f^{(5)}(1/2) = 360$$

Now, eqn (1) becomes,

$$\oint \frac{z^6}{(2z-1)^6} dz = \frac{2\pi i}{64 \times 120} \times 360 = \frac{3\pi i}{32}$$

#

Again. Statement: Let $f(z)$ be analytic funcⁿ inside and on a simple closed curve C enclosing a simply connected region R and if z_0 is any point on the domain of C . Then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n \text{ is a positive integer.}$$

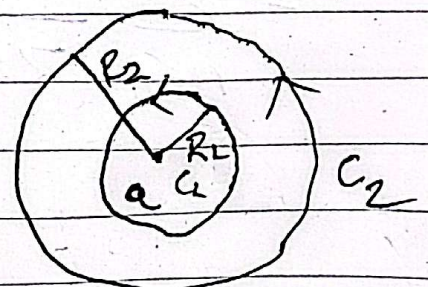
2.9] State Laurent's theorem. Find the Laurent's series for $f(z) = \frac{1}{(z-z^3)}$ in $1 < |z+1| < 2$

$\Rightarrow$ Statement: Let C_1, C_2 be the two concentric circles $|z-a| = r_1$ and $|z-a| = r_2$ where $r_1 < r_2$. Let $f(z)$ be analytic on C_1 and C_2 and in annular region R between them. Then for any point z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

Where.

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz, \quad b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{(z-a)^{-n+1}} dz$$



Given region is $1 < |z+1| < 2$

$$\Rightarrow \left| \frac{1}{z+1} \right| < 1 \text{ and } \left| \frac{z+1}{2} \right| < 1.$$

Let $f(z) = \frac{1}{z-z^3}$

$$= \frac{1}{z(1-z)(1+z)} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{1+z}$$

$$\therefore A=1, B=\frac{1}{2}, C=-\frac{1}{2}$$

$$= \frac{1}{z} + \frac{1}{2} \frac{1}{1-z} - \frac{1}{2} \left(\frac{1}{1+z} \right)$$

$$= \frac{1}{z+1-1} + \frac{1}{2} \left(\frac{1}{1-z-1+1} \right) - \frac{1}{2} \left(\frac{1}{1+z} \right)$$

$$= \frac{1}{(z+1) \left(1 - \frac{1}{z+1} \right)} + \frac{1}{2 \cdot 2} \left(\frac{1}{1 - \left(\frac{z+1}{2} \right)} \right) - \frac{1}{2} \left(\frac{1}{1+z} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(z+1)^{n+1}} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{2^{n+2}} - \frac{1}{2} \frac{1}{1+z}$$

Q. find the image of infinite strip $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

⇒ Given that,

$$f(z) = w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

Comparing with $f(z) = u+iv$ then,

$$u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}$$

Exchange the variable, then,

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

Here, $\frac{1}{4} < y < \frac{1}{2}$ under $w = \frac{1}{z}$ then the transformation will be.

in z -plane	$y = 1/4$	$y = 1/2$
in w -plane	$v = \frac{u^2+v^2}{(-1)}$	$v = \frac{u^2+v^2}{(-2)}$

Since, $-4v = u^2 + v^2$

$$u^2 + v^2 + 4v = 0$$

$$u^2 + (v+2)^2 = 4$$

Circle
~~is a circle~~

which is ~~center~~ of
 Centre, $(0, -2)$, radius
 2.

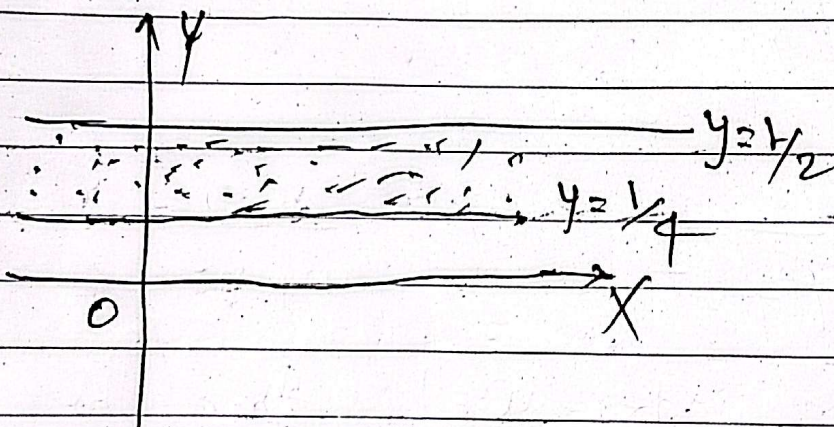
Again,

$$u^2 + v^2 + 2v = 0$$

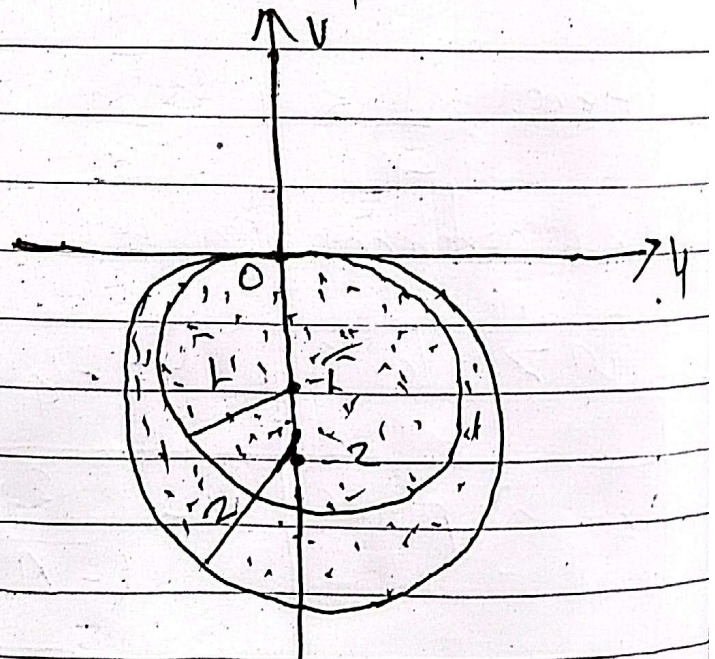
$$u^2 + (v+1)^2 = 1$$

which is also circle
 of Centre $(0, -1)$ radius
 1.

z-plane



w-plane



2.6] Define Singularity, zeros, and poles of a function. Evaluate $\oint_C f(z) dz$ where.

$$f(z) = \frac{e^{2z}}{(z+1)^3}$$

Where C is the ellipse

$$4x^2 + 9y^2 = 16$$

⇒ Singularity: A point $z = z_0$ at which a function $f(z)$ fails to be analytic is called the singular points a singularity of $f(z)$.

Ex. $f(z) = \frac{z}{z-3}$

Here, $z=3$ is a singularity of $f(z)$

Zeros: If a function $f(z)$ is analytic in a region R and is zero at point $z = z_0$ in R then z_0 is called zero of $f(z)$.

Poles: If we can find a positive integer n such that $\lim_{z \rightarrow a} (z-a)^n f(z) \neq 0$ then $z=a$ is called pole of order n for $f(z)$.

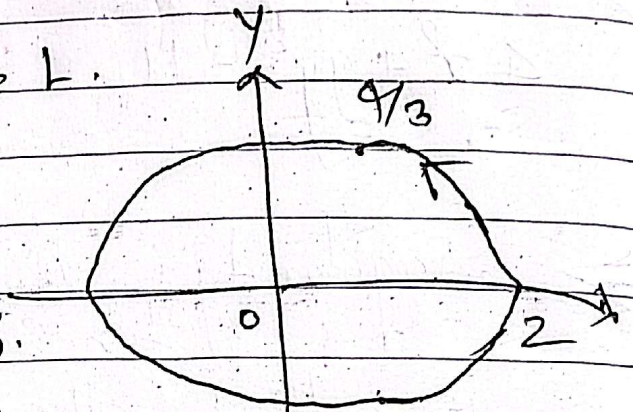
Hen, $\oint \frac{e^{2z}}{(z+1)^3} dz.$

$$C: 4x^2 + 9y^2 = 16$$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{16/9} = 1.$$

clearly,

$z = -1$ of order 3.



Which curve is an ellipse and lies in $z = -1$ of order 3.

Now, the residue of $f(z)$ at point $z_0 = -1$

$$\text{Res.}[f(z), -1] = \lim_{z \rightarrow -1} \frac{1}{2!} \frac{d^2}{dz^2} [e^{2z}]$$

$$= \frac{1}{2} \lim_{z \rightarrow -1} \frac{d}{dz} [2e^{2z}]$$

$$= \frac{1}{2} \lim_{z \rightarrow -1} (4e^{2z})$$

$$= 2e^{-2} = \frac{2}{e^2}$$

Now, By Cauchy residue theorem,

$$\oint_C \frac{e^{2z}}{(z+1)^3} dz = 2\pi i \times \frac{2}{e^2} = \frac{4\pi i}{e^2}$$

#

3a]. State and prove initial theorem and final inverse z-transform of $f(z) = \frac{z^2 - 3z}{(z-5)(z+2)}$

$\Rightarrow$ Statement: If $\mathcal{Z}[f(t)] = \bar{f}(z)$ then $\lim_{t \rightarrow 0} f(t) = \lim_{z \rightarrow \infty} \bar{f}(z)$.

Proof: By definition,

$$\mathcal{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$f(z) = f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \dots$$

Taking limit as $z \rightarrow \infty$ on both sides,

$$\lim_{z \rightarrow \infty} f(z) = f(0) = \lim_{t \rightarrow 0} f(t)$$

Proved

Given $f(z) = \frac{z^2 - 3z}{(z-5)(z+2)}$

then, $z^{n-1} f(z) = \frac{z^{n+1} - 3z^n}{(z-5)(z+2)}$ — (i)

Clearly, $z^{n-1} f(z)$ has poles at $z=5, -2$

We have,

$z^{-1} \{f(z)\} = \text{Sum of residue of } z^{n-1} f(z)$

Here,

residue at $z=5$ is,

$\text{Res.}[f(z), 5] = \lim_{z \rightarrow 5} \frac{z^n (z-3)}{(z+2)} = \frac{5^n \cdot 2}{7} = \frac{2 \cdot 5^n}{7}$

Again,

Residue at $z=-2$ is

$\text{Res.}[f(z), -2] = \lim_{z \rightarrow -2} \frac{z^n (z-3)}{(z-5)} = \frac{(-2)^n (-5)}{-7} = \frac{5(-2)^n}{7}$

Now eqn (i) becomes,

$z^{-1} \{f(z)\} = \frac{2 \cdot 5^n}{7} - \frac{5(-2)^n}{7} = \frac{1}{7} [2 \cdot 5^n - 5(-2)^n]$

#

3.6] Use z-transformation to solve
 $y_{n+2} + by_{n+1} + ay_n = z^n$ given $y_0 = y_1 = 0$

⇒ Given,

$$y_{n+2} + by_{n+1} + ay_n = z^n, \quad y_0 = y_1 = 0$$

Taking z-transform on both sides. ——— (i)

$$z(y_{n+2}) + b[z(y_{n+1})] + a z(y_n) = z(z^n)$$

$$\Rightarrow z^2(\bar{y} - y_0 - \frac{y_1}{z}) + b z(\bar{y} - y_0) + a \bar{y} = \frac{z}{z-2}$$

$$\Rightarrow z^2 \bar{y} + b z \bar{y} + a \bar{y} = \frac{z}{z-2}$$

$$\Rightarrow \bar{y} = \frac{z}{(z-2)(z^2 + bz + a)}$$

$$= \frac{z}{(z-2)(z+3)(z+3)} = \frac{z}{(z-2)(z+3)^2}$$

$$z = \frac{Az}{(z-2)} + \frac{Bz}{(z+3)} + \frac{Cz}{(z+3)^2}$$

———— (ii)

$$A = \frac{1}{25}, C = -\frac{4}{5},$$

Take or Equating Coefficient, z^3 .

$$0 = A + B \Rightarrow A = -B$$

$$\therefore B = -\frac{1}{25}$$

from eqⁿ (ii),

$$Y = \frac{1}{25} \left(\frac{z}{z-2} \right) - \frac{1}{25} \left(\frac{z}{z+3} \right) - \frac{1}{5} \left(\frac{z}{(z+3)^2} \right)$$

Taking z^{-1} on both sides,

$$Y_n = \frac{1}{25} (2^n) - \frac{1}{25} (-3)^n - \frac{1}{5} n (-3)^{n-1}$$

#.

4.a] Define fourier integral. By choosing a suitable function show that.

$$\int_0^{\infty} \frac{\cos \omega x + \omega \sin \omega x}{1 + \omega^2} d\omega = \begin{cases} 0 & \text{if } x < 0 \\ \pi & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

⇒ Fourier integral: let $f(x)$ be a funcⁿ of period $2L$ then fourier integral is,

$$f(x) = \int_0^{\infty} (A(\omega) \cos \omega x + B(\omega) \sin \omega x) d\omega \quad \text{--- (i)}$$

Where, $A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t dt$.

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t dt$$

Here, let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \pi & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases} = \int_0^{\infty} \frac{\cos \omega x + \omega \sin \omega x}{1 + \omega^2} d\omega$

Then, By defⁿ,

find, $A(\omega) = \frac{1}{\pi} \int_0^{\infty} \pi e^{-t} \cos \omega t \, dt$

$$= \left[\frac{e^{-t}}{1+\omega^2} (-\cos \omega t + \omega \sin \omega t) \right]_0^{\infty}$$

$$= \frac{1}{1+\omega^2}$$

$$B(\omega) = \frac{1}{\pi} \int_0^{\infty} \pi e^{-t} \sin \omega t \, dt$$

$$= \frac{\omega}{1+\omega^2}$$

Now, $f(x) = \int_0^{\infty} \frac{\cos \omega x + \omega \sin \omega x}{1+\omega^2} d\omega$ from eqn (1).

Hence proved.

4. b] Find the Fourier transform of the function $f(x) = e^{-x^2/2}$

$\Rightarrow$ Given $f(x) = e^{-x^2/2}$ — (1)

Now, Fourier transform of (i) is,

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} + i\omega x\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 + 2i\omega x + (i\omega)^2)} e^{\frac{(i\omega)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x+i\omega}{\sqrt{2}}\right)^2} dx$$

Put $\left(\frac{x+i\omega}{\sqrt{2}}\right) = p$ then, $dx = \sqrt{2} dp$ then,

$$f(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} \sqrt{2} \int_{-\infty}^{\infty} e^{-p^2} dp$$

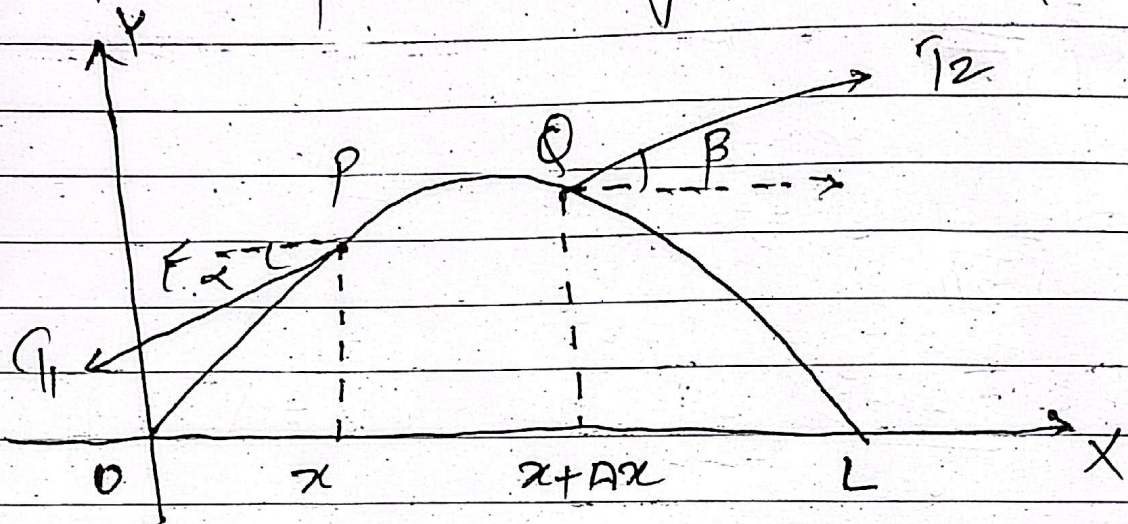
$$= \frac{1}{\sqrt{\pi}} e^{-\omega^2/2} \left[\because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \right]$$

$$= e^{-\omega^2/2}$$

∴

5.a] Derive one dimensional wave eqⁿ with required assumption.

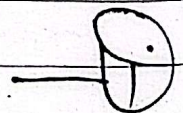
⇒ Statement: let us suppose that the force acting on small portion of string. Since the string does not alter to bend, the tension caused is tangential to the curve of the string at each point.



let us consider, T_1 and T_2 be the tension at ends of points P and Q. The point of string move vertically. so, there is no motion horizontal direction.

∴ The horizontal component of tension must be constant

i.e $T_1 \cos \alpha = T_2 \cos \beta = T(\text{say}) = \text{constant}$



But in Vertical direction,

$-T_1 \sin \alpha$ and $T_2 \sin \beta$ of T_1 and T_2 .

(-ve) sign indicates component P is directed downwards.

By Newton's second law $\therefore$ we get.

$$\text{i.e. } T_2 \sin \beta - T_1 \sin \alpha = \int \Delta x \times \frac{\partial^2 u}{\partial t^2} \quad \textcircled{ii}$$

Where ρ be the mass of deflected string per unit length.

$\Delta x \rightarrow$ be the length of portion of the undeformed string.

Dividing eqⁿ $\textcircled{ii}$ by $\textcircled{i}$ we get,

$$\frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\tan \beta - \tan \alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} \quad \textcircled{iii}$$

Since, $\tan \beta$ is a slope of Q and $\tan \alpha$ is the slope at P as.

$$\tan \alpha = \frac{\partial y}{\partial x} \bigg|_{\alpha = x}$$

$$\tan \beta = \frac{\partial y}{\partial x} \bigg|_{\alpha = x + \Delta x}$$

from (41),

$$\left[\frac{\partial y}{\partial x} \bigg|_{\alpha = x + \Delta x} - \frac{\partial y}{\partial x} \bigg|_{\alpha = x} \right] = \frac{s \Delta x}{T} \frac{\partial^2 y}{\partial x^2}$$

Dividing on both sides by Δx and taking $\lim_{\Delta x \rightarrow 0}$ then,

$$\lim_{\Delta x \rightarrow 0} \left[\frac{\left(\frac{\partial y}{\partial x} \right)_{\alpha = x + \Delta x} - \left(\frac{\partial y}{\partial x} \right)_{\alpha = x}}{\Delta x} \right] = \frac{s}{T} \frac{\partial^2 y}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{s}{T} \frac{\partial^2 y}{\partial x^2}$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{T}{s} \frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (A)}$$

$$\text{Where, } c^2 = T/s$$

This is the required one Dim. Wave eqn.

Q1, Define ^{Partial} differential equation. By separating variables solve.

i) $4xy - 4 = 0$

ii) $2xy + 2y = 0$

$\Rightarrow$ Partial differential equation: A eqⁿ involving one or more partial derivatives of an (unknown) funcⁿ of two or more independent variable is called partial differential eqⁿ.

For Example: one dimen. wave eqⁿ: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

i) Given, $4xy - 4 = 0$

Assume solⁿ of the form,

$$u(x, y) = X \cdot Y$$

Compute

$$4xy = \frac{\partial^2 y}{\partial x \partial y} = X' Y'$$

Substitute,

$$X' Y' - X Y = 0 \Rightarrow X' Y' = X Y$$

$$\frac{X'}{X} = \frac{Y'}{Y} = 1 \text{ (say)}$$

$$X' = kX$$

$$Y' = MY$$

Since: $k = \lambda$ & $M = \frac{1}{\lambda}$. Then, $k = \lambda$

$$M = \frac{1}{\lambda}$$

$$X' = \lambda X \Rightarrow X = Ae^{\lambda x}$$

$$Y' = \frac{1}{\lambda} Y \Rightarrow Y = Be^{y/\lambda}$$

General soln :-

$$u = X \cdot Y = AB e^{\lambda x} \cdot e^{y/\lambda} = Ce^{\lambda x + y/\lambda}$$

Where, $C = AB$

ii) $2xy + 2y^2 = 0$

$\Rightarrow$ let $u = X \cdot Y$

$$u_{xy} = X'Y' \Rightarrow 2X'Y' + 2YX'' = 0$$

Divide both side by $X \cdot Y$

$$2 \frac{X'}{X} \cdot \frac{Y'}{Y} + 2Y = 0 \Rightarrow \frac{X'}{X} \cdot \frac{Y'}{Y} = -Y$$

left hand side is,

$$\frac{X'}{X} = \lambda, \text{ a constant}$$

$$\lambda \cdot \frac{Y'}{Y} = -y \Rightarrow \frac{Y'}{Y} = -\frac{y}{\lambda} \Rightarrow \frac{dY}{Y} = -\frac{y}{\lambda} dy$$

Integrate on both sides,

$$\ln Y = -\frac{y^2}{2\lambda} + C \Rightarrow Y = B e^{-y^2/2\lambda}$$

and X ,

$$\frac{X'}{X} = \lambda \Rightarrow X = A e^{\lambda x}$$

$$\begin{aligned} \text{General soln: } y = X \cdot Y &= A B e^{\lambda x} e^{-y^2/2\lambda} \\ &= C e^{\lambda x - y^2/2\lambda} \end{aligned}$$

Where $C = AB$

#

5.6] Find the temp^t distribution in a laterally insulated thin Copper bar ($c^2 = 1.158 \text{ cm}^2/\text{sec}$) 100cm long and of constant cross section whose end points at $x=0$ and $x=100$ are kept at 0°C and its initial temp^t is $f(x) = \sin(0.01)\pi \cdot x$.

$\Rightarrow$ Given that,

$$c^2 = 1.158 \text{ cm}^2/\text{sec}.$$

$$L = 100 \text{ cm}$$

$$u(0, t) = 0 = u(100, t) \text{ then,}$$

$$\lambda_n = \frac{n\pi}{L} \Rightarrow \lambda_n^2 = \frac{1.158}{100} n^2 \pi^2$$

Given,

$$f(x) = \sin(0.01)\pi x = u(x, 0) \quad \text{--- (i)}$$

The required temp^t is given by

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x e^{-\lambda_n^2 t} \quad \text{--- (ii)}$$

Where,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx.$$

put $L=100$ and $t=0$ eqⁿ (i) then,

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin(0.01)n\pi x \quad \text{--- (ii)}$$

from eqn (i) and (iii)

$$\begin{aligned} \sin(0.01)\pi x &= \sum_{n=1}^{\infty} B_n \sin(0.01)n\pi x \\ &= B_1 \sin(0.01)\pi x + B_2 \sin(0.02)\pi x + \dots \end{aligned}$$

Equating the coefficient,

$$B_1 = 1, \quad B_2 = B_3 = \dots = 0$$

Now,

eqn (ii) becomes

$$y(x,t) = \sin(0.01)\pi x e^{-\left(1.158/100\right)^2 t^2}$$

#

6.a] Find the temp^r in a laterally insulated bar of length L whose ends are kept in at a zero temp^r. assuming that the initial temp^r is,

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{L}{2} \\ L-x & \text{if } \frac{L}{2} \leq x \leq L \end{cases}$$

⇒ Given.

$$\text{Length} = L$$

$$\text{Initial deflection } u(x, 0) = f(x) = \begin{cases} x & \text{if } 0 < x < \frac{L}{2} \\ L-x & \text{if } \frac{L}{2} < x < L \end{cases}$$

$$u(0, t) = 0 = u(L, t)$$

Also, $\lambda_n = \frac{n\pi}{L}$

The solⁿ of one dim. heat eqⁿ is,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x e^{-\lambda_n^2 t} \quad \text{--- (1)}$$

Where,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \cdot dx$$

$$= \frac{2}{L} \left[\int_0^{L/2} x \sin \frac{n\pi}{L} x \cdot dx + \int_{L/2}^L (L-x) \sin \frac{n\pi}{L} x \cdot dx \right]$$

$$= \frac{2}{L} \left[\frac{x(-\cos \frac{n\pi}{L} x)}{(\frac{n\pi}{L})} + \frac{\sin \frac{n\pi}{L} x}{(\frac{n\pi}{L})^2} \right]_0^{L/2} + \left[\frac{(L-x)(-\cos \frac{n\pi}{L} x)}{(\frac{n\pi}{L})} + \frac{(-1)}{(\frac{n\pi}{L})^2} \right]_{L/2}^L$$

$$\frac{\sin \frac{n\pi}{L} x}{(\frac{n\pi}{L})^2} \Big|_{L/2}^L$$

$$= \frac{2}{L} \left[\frac{-L^2}{2n\pi} \left(\cos \frac{n\pi}{2} \right) + \frac{L^2}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right) + \frac{L^2}{2n\pi} \left(\cos \frac{n\pi}{2} \right) + \frac{L^2}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right) \right]$$

$$= \frac{4L}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right)$$

then eqⁿ (i) becomes,

$$y(x, t) = \sum_{n=1}^{\infty} \frac{4L}{n^2\pi^2} \sin \left(\frac{n\pi}{2} \right) \cdot \sin \frac{n\pi}{L} x \cdot e^{-\lambda_n^2 t}$$

#

6.6] Express the Laplacian $\nabla^2 y = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$ in polar coordinate.

$\Rightarrow$ We have Laplacian in Cartesian coordinate is,

$$\nabla^2 y = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \quad \text{--- (i)}$$

We have to change in polar coordinate.
Put $x = r \cos \theta$ and $y = r \sin \theta$.

then, $u_x = \frac{\partial u}{\partial x} = u_r \cdot r_x + u_\theta \theta_x$

diff. w. r. to x,

$$u_{xx} = (u_r r_x)_x + (u_\theta \theta_x)_x$$

$$= (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x \theta_x + u_\theta \theta_{xx}$$

$$= (u_{rr} r_x + u_{r\theta} \theta_x) r_x + u_r r_{xx} +$$

$$(u_{\theta r} r_x + u_{\theta\theta} \theta_x) \theta_x + u_\theta \theta_{xx} - \textcircled{\text{ii}}$$

We have, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}$$

and $\theta_x = \frac{1}{1 + (y/x)^2} \left(-\frac{y}{x^2} \right) = -\frac{y}{r^2}$

$$r_{xx} = \frac{r - x r_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3}$$

and

$$\theta_{xx} = -y \left(\frac{-2}{r^3} \right) r_x = \frac{2xy}{r^4}$$

Putting these value in eqⁿ (i),

$$u_{xx} = \frac{x^2}{r^2} u_{rr} - \frac{2xy}{r^3} u_{r\theta} + \frac{y^2}{r^4} u_{\theta\theta} + \frac{y^2}{r^3} u_r + \frac{2xy}{r^4} u_\theta \quad \text{--- (iii)}$$

$$u_{yy} = \frac{y^2}{r^2} u_{rr} + \frac{2xy}{r^3} u_{r\theta} + \frac{x^2}{r^4} u_{\theta\theta} + \frac{x^2}{r^3} u_r - \frac{2xy}{r^4} u_\theta \quad \text{--- (iv)}$$

from eqⁿ (iii) and (iv),

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

This is said to be Laplacian of u in polar coordinate.

7. Write short note on [Any two].

a) Show that $z\bar{z}$ is not an analytic funcⁿ.

$$\begin{aligned}\Rightarrow \text{Here, } f(z) &= z\bar{z} \\ &= (x+iy)(x-iy) \\ &= (x+iy) \cdot (x-iy) \\ &= x^2 - \cancel{ixy} + \cancel{ixy} - (-y^2) \\ &= x^2 + y^2 + i0\end{aligned}$$

Equating $f(z) = u + iV$ then,

$$u = x^2 + y^2, \quad V = 0$$

By - C.R. eqⁿ,

$$u_x = 2x \quad u_y = 2y \quad v_x = 0, \quad v_y = 0$$

then, $u_x \neq v_y$ and $u_y \neq -v_x$ then

the function is not analytic.

b) Find z-transform of $\sin\left(\frac{n\pi}{2}\right)$ and $\cos\left(\frac{n\pi}{2}\right)$.

$\Rightarrow$ We know, $Z(a^n) = \frac{z}{(z-a)}$

Thus,

$$Z\left(e^{i n \pi / 2}\right) = Z\left[\left(e^{i \pi / 2}\right)^n\right]$$

$$= \frac{z}{z - e^{i \pi / 2}}$$

$$\Rightarrow Z\left(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}\right) = \frac{z}{z - \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)}$$

$$= \frac{z}{z - i}$$

$$= \frac{z^2 - iz}{z^2 + 1}$$

$$= \frac{z^2}{z^2 + 1} - i \frac{z}{z^2 + 1}$$

Equating,

$$Z\left(\cos \frac{n\pi}{2}\right) = \frac{z^2}{z^2 + 1}$$

$$Z\left(\sin \frac{n\pi}{2}\right) = \frac{z}{z^2 + 1}$$

||

C. Solve $y'' - 4y' = 0$

$\Rightarrow$ It's auxiliary eqⁿ is,

$$m^2 - m = 0$$

$$\Rightarrow m = 0, 1$$

$\therefore$ required solⁿ:

$$y = A + Be^x$$

#

Set-3

Year: 2021

Sem: Spring

1. a] Define harmonic function. Prove that the funcⁿ $V = \arg z$ is harmonic. Also, find the ~~its~~ conjugate and the corresponding analytic funcⁿ.

⇒ harmonic function: A real funcⁿ of two variable x and y that posses continuous second order partial derivatives and that satisfies Laplace eqⁿ is called harmonic function.

Here. Given. $V = \arg z = \theta = \tan^{-1}(y/x)$

$V = \tan^{-1}(y/x)$
Diff. w. r. x and y , we get,

$$V_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$V_{xx} = \frac{-y}{(x^2 + y^2)^2} \cdot 2x$$

$$v_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$v_{yy} = \frac{-x \cdot 2y}{(x^2 + y^2)^2}$$

Here,

$$v_{xx} + v_{yy} = 0.$$

Thus v is harmonic funcⁿ.

To find Conjugate.

$$\text{Here, } v_x = \frac{-y}{x^2 + y^2}$$

$$-u_y = \frac{-y}{x^2 + y^2} \quad \left[\because -u_y = v_x \right]$$

Integrate,

$$u = \frac{1}{2} \log(x^2 + y^2) + h(x)$$

Again diff. w. r. to x ,

$$u_x = \frac{x}{x^2 + y^2} + h'(x)$$

$$\frac{x}{x^2+y^2} = \frac{x}{x^2+y^2} + h'(x) \quad [\because u_x = v_y]$$

$$h'(x) = 0$$

integrate, $h(x) = C$ then,

The required harmonic conjugate of v is,

$$u = \frac{1}{2} \log(x^2+y^2) + C \quad \text{Where,}$$

C is complex constant

1.6] State Cauchy's integral formula. and using it integrate $\oint_C \left(\frac{z^2}{z^2-1} \right) dz$. Where C is the circle $|z+i|=1$ in counterclockwise.

⇒ Statement: Let $f(z)$ be analytic in a simply connected domain D . Then for any point z_0 in D and simple closed path C in D that enclosed z_0 . then,

$$\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0).$$

we have:

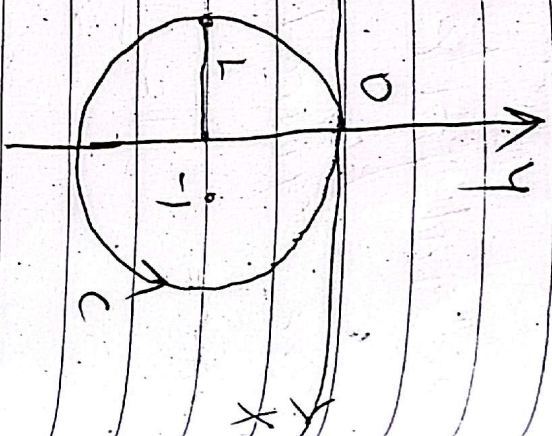
$$f(z) = \frac{z^2}{z^4 - 1} = \frac{z^2}{(z-1)(z+1)(z+i)(z-i)}$$

This is not analytical.

at $z = 1, -1, -i, i$

$$C: |z+i| = 1$$

Here figure we get,



$1, -1, i$ does not lie within the circle and on C .
but $-i$ lies within in C .

$$\text{then, } \oint_C \frac{z^2}{(z^4 - 1)} dz = \left[\frac{z^2}{(z-1)(z+1)(z+i)} \right]_{z=-i} dz.$$

$$\text{By comparing, } f(z) = \frac{z^2}{(z-1)(z+1)(z-i)}$$

By Cauchy integral formula,

$$\oint_C \left(\frac{z^2}{z^4 - 1} \right) dz = 2\pi i f(-i) = 2\pi i \times \frac{-1}{-i(-2i)} = -\pi/2$$

2.a] Find the image of triangular region of the z -plane bounded by the lines $x=0$, $y=0$ & $\sqrt{3}x + y = 1$ under the transformation $w = e^{i(\pi/3)}z$

$\Rightarrow$ Given, that,

$$\begin{aligned} f(z) = w &= e^{i(\pi/3)} \cdot z \\ &= z \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \\ &= (x + iy) \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\ &= \frac{x}{2} + i \frac{\sqrt{3}x}{2} + i \frac{y}{2} - \frac{\sqrt{3}y}{2} \\ &= \frac{\sqrt{3}y + x}{2} + i \left(\frac{\sqrt{3}x + y}{2} \right) \end{aligned}$$

Comparing $f(z) = u + iv$,

$$u = \frac{\sqrt{3}y + x}{2}, \quad v = \frac{\sqrt{3}x + y}{2}$$

exchange variable,

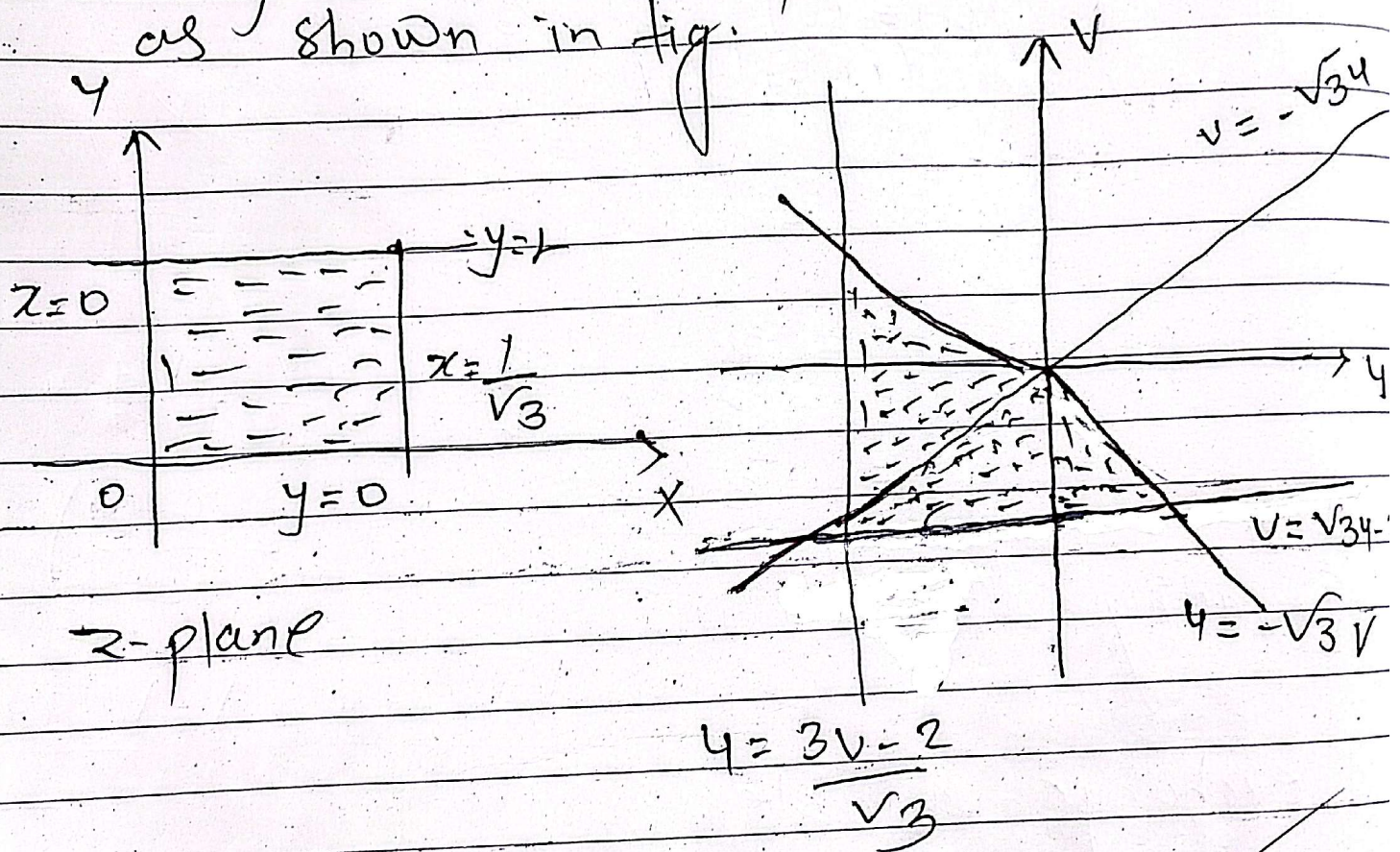
$$x = \frac{u + \sqrt{3}v}{2}, \quad y = \frac{\sqrt{3}u + v}{2}$$

here, $x=0$, $y=0$, $y=1$ & $x=\frac{1}{\sqrt{3}}$

Now,

in z-plane	$x=0$	$x=1/\sqrt{3}$	$y=0$	$y=1$
in w-plane	$u=-\sqrt{3}v$	$u=\frac{3v-2}{\sqrt{3}}$	$u=-\sqrt{3}v$	$u=\sqrt{3}v-2$

The region z-plane is mapped into w-plane as shown in fig.



2.6] Define Singularities of a function $f(z)$.
Find the residue of $f(z) = \frac{z+2}{(z+1)(z^2+1)^2}$

⇒ Singularities of a function $f(z)$: A point $z = z_0$ at which a function $f(z)$ fails to be analytic is called the singular points a singularity of $f(z)$.

Here, we have, $f(z) = \frac{z+2}{(z+1)(z^2+1)^2}$

Thus $z = -1$, and $z = \pm i$ are poles of $f(z)$
where $z = -1$ is simple pole, and $z = \pm i$ are poles of order two.

Here residue at $z = -1$ is,

$$= \lim_{z \rightarrow -1} \left[\cancel{(z+1)} \frac{(z+2)}{\cancel{(z+1)}(z^2+1)^2} \right]$$

$$= \frac{1}{4}$$

Residues at $z = i$ is

$$= \lim_{z \rightarrow i} \frac{d}{dz} \left[\frac{(z-i)^2 (z+2)}{(z+1)(z-i)^2 (z+i)^2} \right]$$

$$= \lim_{z \rightarrow i} \frac{[(z+1)(z+i)^2]^1 (z+2) - (z+1)(z+i)^2 \cdot 1}{[(z+1)(z+i)^2]^2}$$

$$= \frac{-3i}{16}$$

Similarly

$$z = -i,$$

$$\text{Residue at } = \frac{3i}{16}$$

Hence,

required residues are $\frac{1}{4}, \frac{-3i}{16}, \frac{3i}{16}$

#

3. a] state and prove second shifting theorem of z-transform. find z-transform of e^{iat} and hence find $z(\cos at)$.

⇒ statement: If $z[f(t)] = \hat{f}(z)$ then $z[f(t+1)] = z[\hat{f}(z) - f(0)]$

Proof: We have by defⁿ:-

$$z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$z[f(t+1)] = \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f((n+1)T) z^{-n}$$

Put $n+1 = N$ then,

$$z[f(t+1)] = \sum_{N=1}^{\infty} f(NT) z^{-(N-1)}$$

$$= z \sum_{N=1}^{\infty} f(NT) z^{-N}$$

$$= z \left(\sum_{N=0}^{\infty} f(NT) z^{-N} - f(0) \right) = z[\hat{f}(z) - f(0)]$$

proved.

By defⁿ :

$$\mathcal{Z}(e^{-iat}) = \mathcal{Z}(e^{-iat} \cdot 1)$$

$$= [\mathcal{Z}(1)]_{z \rightarrow ze^{iat}}$$

$$= \left(\frac{z}{z-1} \right)_{z \rightarrow ze^{iat}}$$

$$\mathcal{Z}(e^{-iat}) = \frac{ze^{iat}}{ze^{iat} - 1}$$

$$\mathcal{Z}(e^{-iat}) = \frac{z}{z - e^{iat}}$$

$$\begin{aligned} \Rightarrow \mathcal{Z}(\cos at - i \sin at) &= \frac{z}{z - (\cos at + i \sin at)} \\ &= \mathcal{Z}[(z - \cos at) - i \sin at] \\ &= \frac{(z - \cos at)^2 + \sin^2 at}{(z - \cos at)^2 + \sin^2 at} \end{aligned}$$

$$\mathcal{Z}(\cos at - i \sin at) = \frac{\mathcal{Z}[(z - \cos at) - i \sin at]}{z^2 - 2z \cos at + 1}$$

Equating

$$\mathcal{Z}(\cos at) = \frac{\mathcal{Z}(z - \cos at)}{z^2 - 2z \cos at + 1}$$

⊥

3.6] Use Z-transformation to solve difference Equation

$$y_{n+2} - 3y_{n+1} + 2y_n = 4^n, \quad y_0 = 0, y_1 = 1$$

⇒ Given,

$$y_{n+2} - 3y_{n+1} + 2y_n = 4^n, \quad y_0 = 0, y_1 = 1 \quad \text{--- (i)}$$

Taking Z-transform on both sides,

$$Z(y_{n+2}) - 3Z(y_{n+1}) + 2Z(y_n) = Z(4^n)$$

$$Z^2(\bar{y} - y_0 - \frac{y_1}{Z}) - 3Z(\bar{y} - y_0) + 2\bar{y} = \frac{Z}{Z-4}$$

$$\bar{y}(Z^2 - 3Z + 2) - Z = \frac{Z}{Z-4}$$

$$\bar{y} = \frac{1}{(Z^2 - 3Z + 2)} \left(\frac{Z}{Z-4} + Z \right)$$

$$= \frac{Z^2 - 3Z}{(Z-1)(Z-2)(Z-4)}$$

Taking Z^{-1} transform, we get,

$$y_n = Z^{-1} \left\{ \frac{Z^2 - 3Z}{(Z-1)(Z-2)(Z-4)} \right\} \quad \text{--- (ii)}$$

here, $z^{n-1} f(z)$ has poles at $z=1, 2, 4$ of simple pole.

We have,

residue of $z^{n-1} f(z)$ at $z=1$,

$$\begin{aligned} \text{Resi.}[f(z), 1] &= \lim_{z \rightarrow 1} \left[(z-1) \frac{z^{n+1} - 3z^n}{(z-1)(z-2)(z-4)} \right] \\ &= \frac{1-3}{(-1)(-3)} = \frac{-2}{3} \end{aligned}$$

Residue of $z^{n-1} f(z)$ at $z=2$,

$$\begin{aligned} \text{Resi}[f(z), 2] &= \lim_{z \rightarrow 2} \left[(z-2) \frac{z^{n+1} - 3z^n}{(z-1)(z-2)(z-4)} \right] \\ &= \frac{2^n \cdot 2 - 3 \cdot 2^n}{(1)(-2)} = \frac{2^n(2-3)}{-2} \\ &= 2^{n-1} \end{aligned}$$

Resi. of $z^{n-1} f(z)$ at $z=4$,

$$\begin{aligned} \text{Res.}[f(z), 4] &= \lim_{z \rightarrow 4} \left[(z-4) \frac{z^{n+1} - 3z^n}{(z-1)(z-2)(z-4)} \right] \\ &= \frac{4^n \cdot 4 - 3 \cdot 4^n}{3 \times 2} \end{aligned}$$

$$= \frac{4^n (4-3)}{6} = \frac{4^n}{6}$$

Now,

$$Y_n = \text{Sum of residues } z^{n-1} f(z).$$

$$Y_n = -\frac{2}{3} + z^{n-1} + \frac{4^n}{6}$$

#

Q. a) Using Fourier cosine integral, show that,

$$\int_0^{\infty} \frac{\sin w \cos wx}{w} dw = \begin{cases} \pi/2 & \text{if } 0 \leq x < L \\ \pi/4 & \text{if } x = L \\ 0 & \text{if } x > L \end{cases}$$

$\Rightarrow$ we have to show that,

$$\int_0^{\infty} \frac{\sin w \cos wx}{w} dw = \begin{cases} \pi/2 & \text{if } 0 \leq x < L \\ \pi/4 & \text{if } x = L \\ 0 & \text{if } x > L \end{cases}$$

$$\text{Let, } f(x) = \begin{cases} \pi/2 & \text{if } 0 \leq x < L \\ \pi/4 & \text{if } x = L \\ 0 & \text{if } x > L \end{cases}$$

then the fourier integral is,

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x d\omega \quad \text{--- (i)}$$

Where,

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos \omega t dt$$

$$= \frac{2}{\pi} \left[\int_0^L \frac{\pi}{2} \cos \omega t dt + \int_L^{\infty} 0 \cdot \cos \omega t dt \right]$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} \left[\frac{\sin \omega t}{\omega} \right]_0^L$$

$$= \frac{\sin \omega}{\omega}$$

Now, eqn (i) becomes

$$f(x) = \int_0^{\infty} \frac{\sin \omega \cos \omega x}{\omega} d\omega = \text{R.H.S}$$

proved.

4.b] find the Fourier cosine transform of $f(x) = e^{-mx}$ for $m > 0$. then prove that $\int_0^{\infty} \frac{\cos kx}{1+\omega^2} d\omega = \frac{\pi}{2} e^{-k}$.

$\Rightarrow$ We have, Fourier cosine transform,

$$f_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t dt \quad \text{--- (1)}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-mt} \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-mt}}{m^2 + \omega^2} (-m \cos \omega t + \omega \sin \omega t) \right]_0^{\infty}$$

$$f_c(\omega) = \sqrt{\frac{2}{\pi}} \left(\frac{m}{m^2 + \omega^2} \right)$$

Again, Eqn (1) By cosine integral,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(\omega) \cos \omega x d\omega$$

$$e^{-mx} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left(\frac{m \cos \omega x}{m^2 + \omega^2} \right) d\omega \quad \text{--- (2)}$$

Set, $m=k$, $x=k$ and $\omega=x$ then,

$$e^{-k} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos kx}{1+x^2} dx$$

$$\Rightarrow \int_0^{\infty} \frac{\cos kx}{1+x^2} dx = \frac{\pi}{2} e^{-k}$$

proved

5.a] A tightly stretched string of length L , fixed at its ends, is initially in a position given by $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{L}\right)$

If it is released from rest from this position, find the displacement at any point x at time t .

$\Rightarrow$ Given, length = L

$$\text{initial deflection } f(x) = y(x,0) = y_0 \sin^3\left(\frac{\pi x}{L}\right)$$

$$\text{initial velocity } g(x) = 0$$

$$\text{Also, } \lambda_n = \frac{n\pi}{L}$$

the solⁿ of one dimensional wave eqn is,

$$y(x,t) = \sum_{n=1}^{\infty} (B_n \cos \frac{n\pi}{L} x + B_n^* \sin \frac{n\pi}{L} x) \sin \frac{n\pi}{L} x \quad \text{--- (i)}$$

$$\text{Where, } B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$$

$$\text{and } B_n^* = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x dx = 0 \quad \left[\because g(x) = 0 \right]$$

then eqn (i) becomes

$$y(x,t) = \sum_{n=1}^{\infty} B_n \cos \frac{n\pi}{L} x \sin \frac{n\pi}{L} x \quad \text{--- (ii)}$$

Put $t=0$,

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \quad \text{--- (iii)}$$

We have, $\sin 3A = 3 \sin A - 4 \sin^3 A$

$$\sin^3 A = \frac{3 \sin A - \sin 3A}{4}$$

So,

$$y(x,0) = \frac{4_0}{4} \left(3 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} \right)$$

Eqn (iii) gives

$$\frac{40}{4} \left(3 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} \right) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

$$= B_1 \sin \frac{\pi x}{L} + B_2 \sin \frac{2\pi x}{L} + B_3 \sin \frac{3\pi x}{L} + \dots$$

Equating,

$$B_1 = \frac{340}{4}, \quad B_3 = -\frac{40}{4}$$

$$B_4 = B_2 = B_5 = \dots = 0$$

Then eqⁿ (ii) becomes,

$$y(x, t) = \frac{340}{4} \cos \omega_1 t \sin \frac{\pi x}{L} - \frac{40}{4} \cos \omega_3 t \sin \frac{3\pi x}{L}$$

$$\Rightarrow y(x, t) = \frac{40}{4} \left(3 \cos \omega_1 t \sin \frac{\pi x}{L} - \cos \omega_3 t \sin \frac{3\pi x}{L} \right)$$

#

5.6] Find the temp^r in a laterally insulated bar of length $L = 10\text{cm}$ whose ends are kept at zero temp^r, assuming that

$$\text{initial temp^r is } f(x) = \begin{cases} x & \text{if } 0 < x < \frac{L}{2} \\ L-x & \text{if } \frac{L}{2} < x < L \end{cases}$$

$\Rightarrow$ Given.

$$\text{length } (L) = 10\text{cm}$$

$$\text{initial deflection } y(x, 0) = f(x) = \begin{cases} x & \text{if } 0 < x < \frac{L}{2} \\ L-x & \text{if } \frac{L}{2} < x < L \end{cases}$$

$$\text{And } y(0, t) = 0 = y(L, t)$$

$$\text{Also, } \lambda_n = \frac{n\pi}{L}$$

We have solⁿ of one dim. heat eqⁿ is

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x e^{-\lambda_n^2 t} \quad (1)$$

Where,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$$

$$= \frac{2}{L} \left[\int_0^{L/2} x \sin \frac{n\pi}{L} x dx + \int_{L/2}^L (L-x) \sin \frac{n\pi}{L} x dx \right]$$

$$= \frac{2}{L} \left\{ \left[\frac{-x \cos \frac{n\pi}{L} x}{\left(\frac{n\pi}{L}\right)} + \frac{\sin \frac{n\pi}{L} x}{\left(\frac{n\pi}{L}\right)^2} \right]_0^{L/2} + \left[\frac{(L-x) - \cos \frac{n\pi}{L} x}{\left(\frac{n\pi}{L}\right)} \right]_{L/2}^L \right\}$$

$$= \frac{2}{L} \left[\frac{-L^2 \cos \frac{n\pi}{2}}{2n\pi} + \frac{L^2 \sin \left(\frac{n\pi}{2}\right)}{n^2 \pi^2} + \frac{L^2 \cos \left(\frac{n\pi}{2}\right)}{2n\pi} + \frac{L^2}{n^2 \pi^2} \left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{4L}{n^2 \pi^2} \sin \left(\frac{n\pi}{2}\right)$$

$$= \frac{40}{n^2 \pi^2} \sin \left(\frac{n\pi}{2}\right) \quad [\because L = 10 \text{ cm}]$$

Now, eqn (i) becomes

$$u(x, t) = \sum_{n=1}^{\infty} \frac{40}{n^2 \pi^2} \sin \left(\frac{n\pi}{2}\right) \sin \left(\frac{n\pi}{10}\right) x e^{-\frac{\pi^2 n^2}{100} t}$$

#

6.a] Find the solution of differential equation $y^2 y_x - x^2 y_y = 0$ using separating of variables.

$\Rightarrow$ Given,

$$y^2 y_x - x^2 y_y = 0 \quad \text{--- (i)}$$

Let $y(x, y) = f(x) \cdot G(y)$. --- (ii) be the solⁿ of given differential eqⁿ.
Where f and G are funcⁿ of x and y respectively.

$$\text{Here, } y_x = \frac{\partial y}{\partial x} = f' G$$

$$y_y = f \dot{G}$$

Where dashes and dots denote differential with x to x and y .

eqⁿ (i),

$$y^2 f' G - x^2 f \dot{G} = 0$$

$$\frac{f'}{f x^2} = \frac{\dot{G}}{G y^2} = k \text{ (say)}$$

This gives, $\frac{f'}{f x^2} = k$ --- (iii)

$$\frac{\dot{G}}{Gy^2} = k \quad \text{--- (iv)}$$

from (iii),

$$\frac{F'}{F} = x^2 k$$

Integrating on both side w.r. to x ,

$$\log F = k \frac{x^3}{3} + C$$

$$F = e^{\frac{kx^3}{3}} D \quad \text{Where } D \text{ is Constant.}$$

from (iv),

$$\frac{\dot{G}}{G} = \frac{2}{y} k$$

Integrating on both sides w.r. to y .

$$\log G = k \frac{y^2}{2} + A$$

$$G = e^{\frac{ky^2}{2}} E \quad \text{Where } E \text{ is Constant}$$

∴ The required solⁿ of differential eqⁿ:

$$u(x, y) = D e^{\frac{kx^3}{3}} \cdot E e^{\frac{ky^2}{2}}$$

$$= DE e^{k/3(x^3+y^3)}$$

$$= M e^{k/3(x^3+y^3)}$$

Where k and M are constant.

6.b) Find the solution of one dimensional wave eqⁿ by D'Alembert's method

⇒ We have,

one dime. wave eqⁿ is,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

Where $c^2 = 1/g$

let $V = x + ct$ and $z = x - ct$. Then y is a function of v and z .

$$\therefore y_x = u_v v_x + u_z z_x$$

$$y_x = u_v + u_z$$

and,

$$y_{xx} = (u_v + u_z)_x$$

$$= (u_v + u_x)_v v_x + (u_v + u_z)_z z_x$$

$$= u_{vv} + u_{zv} + u_{vz} + u_{zz}$$

$$u_{xx} = u_{vv} + 2u_{vz} + u_{zz}$$

Similarly,

$$u_{tt} = c^2 (u_{vv} - 2u_{vz} + u_{zz})$$

in eqⁿ (F) becomes,

$$c^2 (u_{vv} - 2u_{vz} + u_{zz}) = c^2 (u_{vv} + 2u_{vz} + u_{zz})$$

$$4u_{vz} = 0$$

$$u_{vz} = \frac{\partial^2 u}{\partial z \partial v} = 0$$

Integrating w.r. to z ,

$$\frac{\partial u}{\partial v} = h(v) \quad \text{--- (H)}$$

where $h(v)$ is an arbitrary function of v .

Again, integrate w.r. to v ,

$$y = \int h(v) dv + \psi(z) \quad \text{--- (iii)}$$

where $\psi(z)$ is an arbitrary function of z .

Also, eqn (iii), the integral is a function of v , say $\phi(v)$. then,

$$y = \phi(v) + \psi(z)$$

$$y(x, t) = \phi(x+ct) + \psi(x-ct) \quad \text{--- (iv)}$$

This is known as D'Alembert's solution of wave eqn.

initial deflection,

$$y(x, 0) = f(x) \quad \text{--- (v)}$$

and initial velocity, $y_t(x, 0) = g(x) \quad \text{--- (vi)}$

Differentiate eqn (iv) w.r. to t ,

$$y_t(x, t) = c\phi'(x+ct) - \psi'(x-ct) \quad \text{--- (vii)}$$

Put $t=0$, in eqn (iv) and (vii),

$$y(x, 0) = \phi(x) + \psi(x) = f(x) \quad \text{--- (viii)}$$

$$y_1(x, 0) = c\phi'(x) - c\psi'(x) = g(x) \quad \text{--- (ix)}$$

$$\Rightarrow c\phi'(x) - c\psi'(x) = g(x)$$

$$\phi'(x) - \psi'(x) = \frac{g(x)}{c}$$

Integrating w.r. to x ,

$$\phi(x) - \psi(x) = K(x_0) + \frac{1}{c} \int_{x_0}^x g(s) ds \quad \text{--- (x)}$$

$$\text{Where, } K(x_0) = \phi(x_0) - \psi(x_0)$$

Adding (viii) and (x) we get,

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_{x_0}^x g(s) ds + \frac{1}{2} K(x_0) \quad \text{--- (xi)}$$

and subtracting (x) from (viii),

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_{x_0}^x g(s) ds - \frac{1}{2} K(x_0) \quad \text{--- (xii)}$$

Now, replace x by $x+ct$ and x by $x-ct$

in eqⁿ (xi) and (xii), and adding them,

$$y(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$y(x,t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

As the initial velocity is zero, then,

$$y(x,t) = \frac{1}{2} [f(x+ct) + f(x-ct)]$$

Which is required solution of one-dim. wave eqn by D'Alembert's. (A)

7. Attempts All question.

7a] check analyticity of $f(z) = z^2$

$$\Rightarrow \text{Given } f(z) = z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 \\ = (x^2 - y^2) + i2xy$$

Equating,

$$f(z) = u + iv, \\ u = x^2 - y^2 \\ v = 2xy$$

$$u_x = 2x$$

$$v_x = 2y$$

$$u_y = -2y$$

$$v_y = 2x$$

By C-R eqⁿ,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

then the given funcⁿ is analytic.

7.6] Find the poles of the function $f(z)$

$$= \frac{\sinh z}{(z - i\pi)}$$

$\Rightarrow$ Here, $f(z)$ is $\sinh z$ in entire function does not introduce any poles

$$z - i\pi = 0$$

$z = i\pi$ Which is required pole or order 1. or (simple pole).

7.1) find z-transform of $z(n^2)$.

$\Rightarrow$ Here,

$$z(n^2) = -z \frac{d}{dz} z(n)$$

$$= -z \frac{d}{dz} \left(\frac{z}{(z-1)^2} \right)$$

$$= -z \frac{(z-1)^2 \cdot 1 - z \cdot 2(z-1)}{(z-1)^4}$$

$$= -z \frac{(z-1) \{ (z-1) - 2z \}}{(z-1)^3}$$

$$= -z \frac{z-1-2z}{(z-1)^3}$$

$$= -z \frac{(-1-z)}{(z-1)^3}$$

$$= \frac{z(z+1)}{(z-1)^3} \quad \#$$

Set-4

Year: 2023
Sem: Fall

1.a] Define harmonic function. check $u = x^3 - 3xy^2$ is harmonic or not? If yes, Find the corresponding harmonic conjugate v of u .

$\Rightarrow$ A real function of two variable x and y that posses continuous. second order partial derivatives and that satisfies Laplace eqn is called harmonic funcⁿ.

Here. Given, $u = x^3 - 3xy^2$

$$u_x = 3x^2 - 3y^2$$

$$u_{xx} = 6x$$

$$u_y = -6xy$$

$$u_{yy} = -6x \quad \text{then,}$$

$u_{xx} + u_{yy} = 0$ hence the u is harmonic.
for harmonic conjugate,

$$u_x = u_y$$

$$u_y = 3x^2 - 3y^2$$

Integrate w.r. to y.

$$u = 3x^2y - y^3 + h(x) \quad \text{--- (i)}$$

Where, $h(x)$ is funcⁿ of x or constant.

Differentiate, eqn (i) w.r. to x .

$$u_x = 6xy + h'(x)$$

$$-6xy = 6xy + h'(x) \quad [\because -u_y = u_x]$$

$$h'(x) = 0$$

integrate,

$$h(x) = C \quad \text{Where } C \text{ is constant}$$

Eqn (i) becomes,

$$u = 3x^2y - y^3 + C \quad \text{is required harmonic conjugate.}$$

1.6] State Cauchy Integral formula for derivatives. Evaluate $\oint_C \frac{z^6}{(2z-1)^6} dz$

Where C is unit circle $|z|=1$, CounterClockwise

$\Rightarrow$ Statement: Let $f(z)$ be analytic funcⁿ inside and on a simple closed curve C enclosing a simply connected region R and if z_0 is any point on the domain of C then,

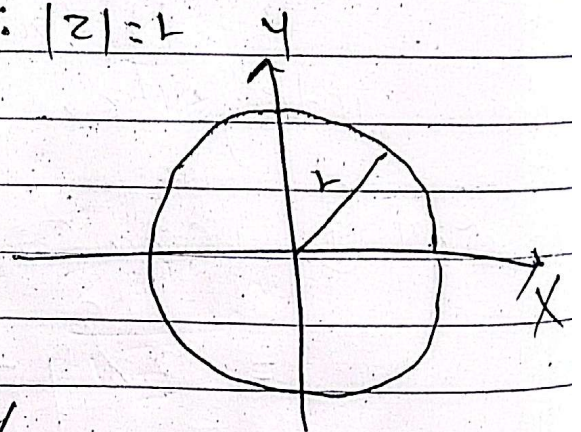
$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n \text{ is a positive integer}$$

Here, $\oint_C \frac{z^6}{(2z-1)^6} dz$ $C: |z|=1$

The given funcⁿ is not analytic at

$$2z-1=0 \Rightarrow z = \frac{1}{2}$$

Which lies inside the circle.



So, $z_0 = 1/2$

By Cauchy Integral formula,

$$\oint \frac{1}{2^6} \frac{z^6}{(2z-1)^6} dz = \frac{1}{2^6} \frac{2\pi i}{5!} f^{(5)}\left(\frac{1}{2}\right) \quad \text{--- (i)}$$

$$f(z) = z^6$$

$$f'(z) = 6z^5$$

$$f''(z) = 30z^4$$

$$f'''(z) = 120z^3$$

$$f^{(4)}(z) = 360z^2$$

$$f^{(5)}(z) = 720z$$

$$f^{(5)}\left(\frac{1}{2}\right) = 720 \times \frac{1}{2} = 360$$

Now, eqⁿ (i),

$$\oint \frac{1}{2^6} \frac{z^6}{(2z-1)^6} = \frac{2\pi i}{64 \times 120} \times 360$$

$$= \frac{3\pi i}{32}$$

$$\#$$

2. [Q] Find the image of triangular region of the z -plane bounded by lines $x=0$, $y=0$, $x+y=1$ under the transformation $w = ze^{i\pi/4}$ and show the sketch in the diagram.

$\Rightarrow$ Here.

$$f(z) = w = ze^{i\pi/4}$$

$$= (x+iy) \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$= (x+iy) \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

$$= \frac{x}{\sqrt{2}} + i \frac{x}{\sqrt{2}} + \frac{iy}{\sqrt{2}} - \frac{y}{\sqrt{2}}$$

$$= \left(\frac{x-y}{\sqrt{2}} \right) + i \left(\frac{x+y}{\sqrt{2}} \right)$$

$$f(z) = u+iv, \text{ Equating,}$$

$$u = \frac{x-y}{\sqrt{2}}$$

$$v = \frac{x+y}{\sqrt{2}}$$

exchange variable

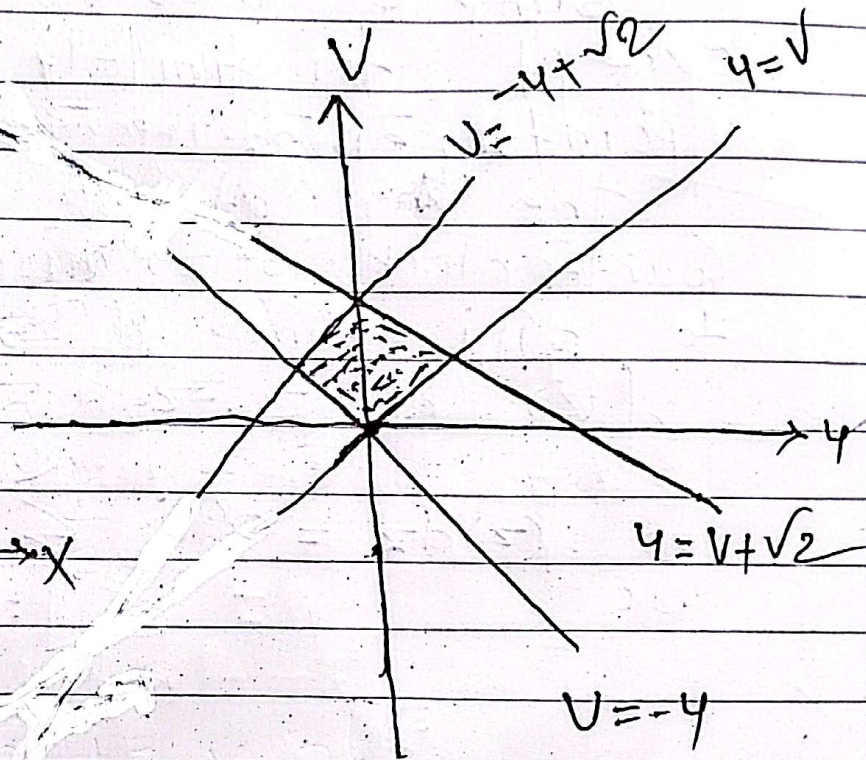
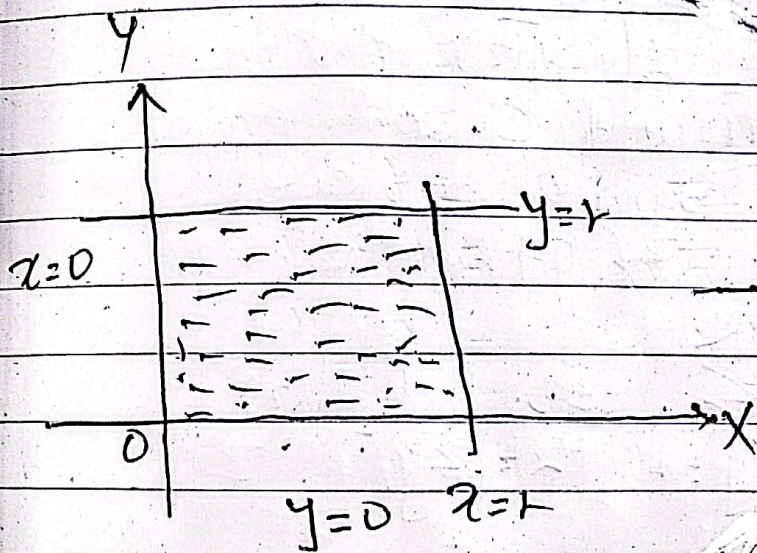
$$x = \frac{u+v}{\sqrt{2}}$$

$$y = \frac{v-u}{\sqrt{2}}$$

z-plane bounded lines: $x=0$, $y=0$, $x=1$, $y=1$.

in z-plane	$x=0$	$x=1$	$y=0$	$y=1$
in w-plane	$u=v$	$u=v+\sqrt{2}$	$v=-u$	$v=-u-\sqrt{2}$

Now, sketch,



z-plane

w-plane

2.6] State Cauchy residue Theorem. By applying Cauchy residue theorem, evaluate $\oint_C \frac{4-3z}{z(z-1)(z-2)} dz$

Where $C: |z| = \frac{3}{2}$

⇒ Statement: let $f(z)$ be analytic inside a simple closed path C and on C except for finitely many singular points $z_1, z_2, \dots, z_k$ in C . Then the integral of $f(z)$ taken counter clockwise around C and equals to $2\pi i$ times of sum of residue of $f(z)$ at $z_1, z_2, \dots, z_k$. That is,

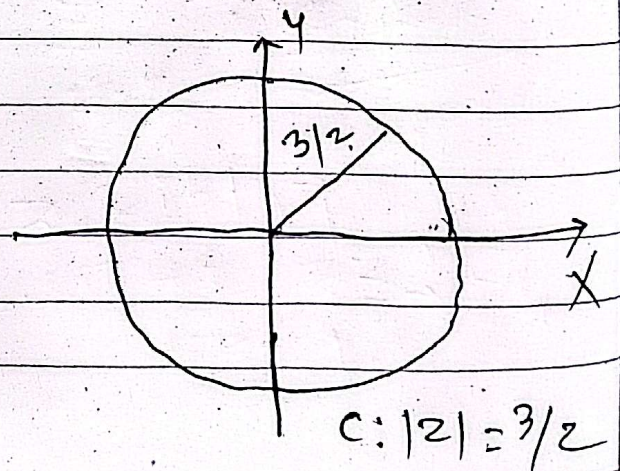
$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^K \text{res. } [f(z)]_{z=z_j}$$

Here $\oint_C \frac{4-3z}{z(z-1)(z-2)} dz$ $C: |z| = 3/2$

funcⁿ has poles at,

$$z(z-1)(z-2) = 0$$

$$\Rightarrow z = 0, 1, 2$$



Since given circle $|z| = 3/2$ with centre $(0,0)$ and radius $3/2$ lies only $z = 0$, a point not $z = 2$. ✓

So, our residue over these point,

The residue at $z = 0$, simple pole,

$$\begin{aligned} \text{Res.}[f(z), 0] &= \lim_{z \rightarrow 0} \left[\cancel{z} \frac{(4-3z)}{\cancel{z}(z-1)(z-2)} \right] \\ &= \frac{4}{(-1)(-2)} = 2 \end{aligned}$$

Residue at $z = 1$, simple pole.

$$\begin{aligned} \text{Res.}[f(z), 1] &= \lim_{z \rightarrow 1} \left[\cancel{(z-1)} \frac{(4-3z)}{z \cancel{(z-1)}(z-2)} \right] \\ &= \frac{4-3}{1 \cdot (-1)} = -1 \end{aligned}$$

Now, By Cauchy residue theorem,

$$\begin{aligned} \oint_C \frac{4-3z}{z(z-1)(z-2)} dz &= 2\pi i [\text{sum of residue}] \\ &= 2\pi i [2-1] = 2\pi i \quad \# \end{aligned}$$

3a) state and prove first shifting theorem.
 of z-transform. Using it evaluate
 the z-transform of $a^n \cos bt$ and
 $a^n \sin bt$.

⇒ statement: If $\mathcal{Z}[f(t)] = F(z)$ then
 $\mathcal{Z}[e^{at} f(t)] = F[ea z]$

Proof: By definition,

$$\mathcal{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad (1)$$

$$\mathcal{Z}[e^{at} f(t)] = \sum_{n=0}^{\infty} f(nT) [ea z]^{-n}$$

$$= \sum_{n=0}^{\infty} [e^{-anT} f(nT) z^{-n}]$$

$$= \mathcal{Z}[e^{at} f(t)]$$

proved

We have,

$$\text{Here, } \mathcal{Z}(a^n e^{ibt}) = \mathcal{Z}(e^{ibt}) \quad z \rightarrow z/a$$

$$= \left(\frac{z}{z - e^{ibt}} \right)_{z \rightarrow z/a}$$

$$= \frac{z/a}{z/a - e^{ibt}}$$

$$= \frac{z}{z - a(\cos bt + i \sin bt)}$$

$$z(a^n \cos bt + i a^n \sin bt) = \frac{z[(z - a \cos bt) - i a \sin bt]}{(z - a \cos bt)^2 + (a \sin bt)^2}$$

$$= \frac{z[(z - a \cos bt) - i a \sin bt]}{z^2 - 2a z \cos bt + a^2}$$

Equating, then

$$z(a^n \cos bt) = \frac{z(z - a \cos bt)}{z^2 - 2a z \cos bt + a^2}$$

$$z(a^n \sin bt) = \frac{z a \sin bt}{z^2 - 2a z \cos bt + a^2}$$

#

3.6] Solve the differential equation by using Z-transform.

$$y_{n+2} - 4y_{n+1} + 4y_n = 2^n \text{ with } y_0 = 0, y_1 = 1$$

$$\Rightarrow \text{Given } y_{n+2} - 4y_{n+1} + 4y_n = 2^n, y_0 = 0, y_1 = 1$$

— (i)

Taking Z-transform on both sides

$$Z(y_{n+2}) - 4Z(y_{n+1}) + 4Z(y_n) = Z(2^n)$$

$$Z^2(\bar{y} - y_0 - \frac{y_1}{Z}) - 4Z(\bar{y} - y_0) + 4\bar{y} = \frac{Z}{(Z-2)}$$

$$\Rightarrow Z^2\bar{y} - (3 - 4Z)\bar{y} + 4\bar{y} = \frac{Z}{(Z-2)}$$

$$\Rightarrow \bar{y} = \frac{1}{(Z^2 - 4Z + 4)} \left(\frac{Z}{Z-2} + \frac{Z}{Z-2} \right)$$

$$= \frac{1}{(Z^2 - 4Z + 4)} \cdot \left(\frac{Z + Z^2 - ZZ}{(Z-2)} \right)$$

$$\bar{y} = \frac{z^2 - z}{(z-2)(z-2)(z-2)} = \frac{z^2 - z}{(z-2)^3}$$

Making z^{-1} on both side:

$$y_n = z^{-1} \left\{ \frac{z^2 - z}{(z-2)^3} \right\} - \text{ii}$$

$$\text{Here } f(z) = \frac{z^2 - z}{(z-2)^3}$$

$z^{n-1} f(z)$ has poles at $z=2$ of order 3.

residue at $z=2$,

$$\text{Residue}[f(z), 2] = \lim_{z \rightarrow 2} \frac{1}{z^2} \frac{d}{dz} \left[\frac{(z-2)^3 (z^{n+1} - z^n)}{(z-2)^3} \right]$$

$$= \lim_{z \rightarrow 2} \frac{1}{z^2} \frac{d}{dz} [z^{n+1} - z^n]$$

$$= \lim_{z \rightarrow 2} \frac{1}{z^2} \frac{d}{dz} [(n+1)z^n - n z^{n-1}]$$

$$= \lim_{z \rightarrow 2} \frac{1}{2} \left[(n+1)n \cdot z^{n-1} - n(n-1)z^{n-2} \right]$$

$$= \frac{1}{2} \left[(n^2 + n)(2)^{n-1} - (n^2 - n)(2)^{n-2} \right]$$

$$= n 2^{n-3} [2n+2 - n+1]$$

$$= n 2^{n-3} [n+3]$$

$$= n(n+3) 2^{n-3}$$

eqⁿ (ii) becomes

$$y_n = n(n+3) 2^{n-3}$$

#

4.a) Show that, $\int_0^{\infty} \frac{\sin \omega \pi \sin x \omega}{1 + \omega^2} d\omega = \begin{cases} \frac{\pi}{2} \sin x & \text{if } 0 \leq x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

>

let $f(x) = \begin{cases} \frac{\pi}{2} \sin x & \text{if } 0 \leq x \leq \pi \\ 0 & \text{if } x > \pi \end{cases}$

(i)

By Fourier sine integral of $f(x)$ is,

$$f(x) = \int_0^{\infty} B(\omega) \sin \omega x \, d\omega \quad \text{--- (ii)}$$

$$\text{Where, } B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \sin \omega t \, dt.$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \frac{\pi}{2} \sin t \sin \omega t \, dt \right]$$

$$= \frac{1}{2} \int_0^{\pi} 2 \sin t \sin \omega t \, dt$$

$$= \frac{1}{2} \left[\int_0^{\pi} \cos(\omega - 1)t - \cos(\omega + 1)t \, dt \right]$$

$$\left[\because 2 \sin A \cos B = \cos(A - B) - \cos(A + B) \right]$$

$$= \frac{1}{2} \left[\frac{\sin(\omega - 1)t}{(\omega - 1)} - \frac{\sin(\omega + 1)t}{(\omega + 1)} \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\frac{\sin \omega \pi \cos \omega \pi}{1 - \omega^2} - \frac{\sin \omega \pi \cos \pi}{1 - \omega^2} \right]$$

$$\left[\because \sin 0 = 0, \sin \pi = 0 \right]$$

$$= \frac{\sin \omega \pi}{\pi} \left[\frac{\cancel{\pi}}{1 - \omega^2} \right]$$

$$= \frac{\sin \omega \pi}{1 - \omega^2}$$

Now, eqⁿ (ii),

$$f(x) = \int_0^{\infty} \frac{\sin \pi \omega \sin \omega x}{1 - \omega^2} d\omega = \text{P.H.L.}$$

proved.

4.6]. Find Fourier sine transform of $f(x) = e^{-x}$ for $x > 0$. The prove that,

$$\int_0^{\infty} \frac{x \sin mx}{1 + x^2} dx = \frac{\pi}{2} e^{-m} \text{ for } m > 0.$$

⇒ By Fourier Sine transform,

$$f_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t dt \quad \text{--- (i)}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t} \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-t}}{1+\omega^2} (-\sin \omega t - \omega \cos \omega t) d\omega$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{1+\omega^2} \right)$$

Again, By fourier sine integral,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\omega) \sin \omega x d\omega \quad \text{--- (ii)}$$

Now,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{\omega}{1+\omega^2} \sin \omega x d\omega$$

$$e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{\omega \sin \omega x}{1+\omega^2} d\omega$$

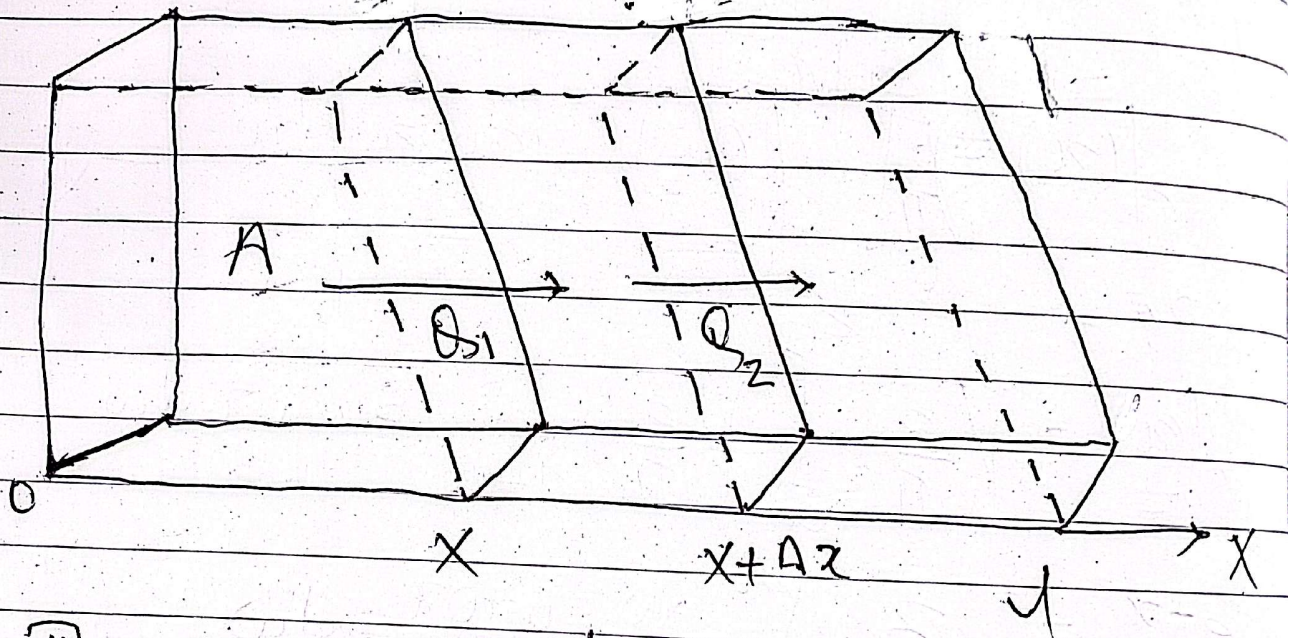
Set $x=m$, $\omega=x$ then,

$$\frac{\pi}{2} e^{-m} = \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx$$

Proved

5.a] Derive one dim. heat eqⁿ $\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t}$ with necessary assumption.

⇒ let us suppose that the flow of heat by conduction of in a uniform bar. Suppose the sides of the bar is insulated.



The amount of heat crossing any section of the bar per seconds depends on the Area A of the cross section. the Conductivity k of the material of the bar.

The temp^r gradient $\frac{\partial \theta}{\partial x}$.

Q_1 , the quantity of heat flowing into the section at distance x .

$$= -KA \left(\frac{\partial y}{\partial x} \right)_x \text{ per sec.}$$

Similarly, -ve sign indicate $\times$ increase, y decrease.

Again, Q_2 at distance, $x + \Delta x$.

$$= -KA \left(\frac{\partial y}{\partial x} \right)_{x+\Delta x} \text{ per sec.}$$

Where, Δx thickness of slab.

Hence,

$$Q_1 - Q_2 = KA \left[\left(\frac{\partial y}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x} \right)_x \right] \text{ per sec.}$$

We have the rate of change of heat in the

$$\text{Slab} = \rho A \Delta x \frac{\partial y}{\partial t}$$

Where, $S \rightarrow$ specific heat
 $\rho \rightarrow$ density of material

from (i) and (ii),

$$SS A \Delta x \frac{\partial y}{\partial t} = kA \left[\left(\frac{\partial y}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x} \right)_x \right]$$

Taking lim on both sides,
 $\Delta x \rightarrow 0$

$$SS \frac{\partial y}{\partial t} = k \left[\frac{\left(\frac{\partial y}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x} \right)_x}{\Delta x} \right]$$

$$SS \frac{\partial y}{\partial t} = k \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial y}{\partial t} = \frac{k}{SS} \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (A)}$$

Where $c^2 = \frac{k}{SS}$ diffusivity
 of material.

5. b] Find $y(x, t)$ from one dimen. wave eqn $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ with boundary Condⁿ $y(0, t) = 0 = y(L, t)$, initial deflection $f(x)$ and initial velocity

$$\left. \frac{\partial y}{\partial t} \right|_{t=0} = g(x).$$

$\Rightarrow$ By one dim. eqn is.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad 0 < x < L, t > 0 \quad \text{--- (i)}$$

$$y(0, t) = 0 = y(L, t)$$

$$y(x, 0) = f(x), \quad \left. \frac{\partial y}{\partial t} \right|_{t=0} = g(x)$$

$$\text{Let } y(x, t) = f(x) \cdot G(t). \quad \text{--- (ii)}$$

$$f \ddot{G} = c^2 f'' G$$

$$\Rightarrow \frac{\ddot{G}}{c^2 G} = \frac{f''}{f} = -\lambda \text{ (say)}$$

$$F'' + \lambda F = 0 \quad (\text{with } F(0) = X(L) = 0)$$

$$G'' + c^2 \lambda G = 0$$

Solving them,

$$f_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n=1, 2, \dots$$

$$G_n(t) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)$$

Now,

General solⁿ :

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

Apply condition,

initial displacement : ∞

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{Where, } A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

initial velocity,

$$\left. \frac{\partial y}{\partial t} \right|_{t=0} = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{Where } B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Now, required solution, $\therefore$

$$y(x,t) = \sum_{n=1}^{\infty} \left[\left(\frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right) \cos\left(\frac{n\pi ct}{L}\right) + \left(\frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \right) \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

#

6a] A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temp and the temp initially is

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 50 \\ 100-x & \text{if } 50 \leq x \leq 100 \end{cases}$$

find the temp distribution on the rod at any time.

$\Rightarrow$ Given length $(l) = 100 \text{ cm}$

initial velocity $g(x) = 0$

initial deflection $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 50 \\ 100-x & \text{if } 50 \leq x \leq 100 \end{cases}$

$$u(x,0) = 0 = u(l,t)$$

Also,

$$T_n = \frac{n\pi c}{L} = \frac{n\pi}{100}$$

The solⁿ of one dime. heat eqⁿ:-

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x e^{-\gamma_n^2 t} \quad \text{--- (i)}$$

Where, $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$.

$$\Rightarrow B_n = \frac{2}{100} \left[\int_0^{50} x \sin \frac{n\pi}{100} x \, dx + \int_{50}^{100} (100-x) \sin \frac{n\pi}{100} x \, dx \right]$$

$$= \frac{1}{50} \left[-x \cos \frac{n\pi}{100} x + \sin \frac{n\pi}{100} x \right]_0^{50} + \left[(100-x) \cos \frac{n\pi}{100} x - \left(\frac{n\pi}{100} \right) \right]_{50}^{100}$$

$$\left. \sin \frac{n\pi}{100} x \right|_0^{100} \left. \left(\frac{n\pi}{100} \right) \right|_0^{100} \left. \left(\frac{n\pi}{100} \right) \right|_0^{100} \left. \left(\frac{n\pi}{100} \right) \right|_0^{100}$$

$$= \frac{1}{50} \left[-50 \cos \left(\frac{n\pi}{2} \right) + \left(\frac{100}{n\pi} \right)^2 \sin \frac{n\pi}{2} + \frac{50 \cos \left(\frac{n\pi}{2} \right)}{n\pi} + \left(\frac{100}{n\pi} \right)^2 \sin \frac{n\pi}{2} \right]$$

$$= \frac{400}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right)$$

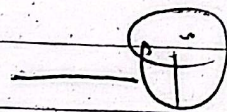
Now, eqⁿ (i) becomes,

$$y(x, t) = \sum_{n=1}^{\infty} \frac{400}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) \cdot \sin\left(\frac{n\pi}{100}x\right) \cdot e^{-\left(\frac{n\pi}{100}\right)^2 t}$$

6.6] Express the Laplacian $\nabla^2 y = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$ in polar coordinates.

⇒ The Laplacian in Cartesian coordinates

$$\nabla^2 y = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$$



to change.

$$x = r \cos \theta \quad y = r \sin \theta$$

then,

$$y_x = \frac{\partial y}{\partial x} = y_r r_x + y_\theta \theta_x$$

Again diff. w.r. to x,

$$y_{xx} = (y_r r_x)_x + (y_\theta \theta_x)_x$$

$$= (y_r)_x r_x + y_r r_{xx} + (y_\theta)_x \theta_x + y_\theta \theta_{xx}$$

$$= (y_{rr} r_x + y_{r\theta} \theta_x) r_x + y_r r_{xx} + (y_{\theta r} r_x + y_{\theta\theta} \theta_x) \theta_x + y_\theta \theta_{xx} \quad \text{--- (iii)}$$

We have.

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}$$

$$\text{and } \theta_x = \frac{1}{1 + (y/x)^2} \cdot \frac{(-y)}{x^2} = \frac{-y}{r^2}$$

$$y_{xx} = \frac{1 \cdot x r_x}{r^2} = \frac{1}{r} - \frac{x^3}{r^3} = \frac{y^2}{r^3}$$

and

$$\theta_{xx} = -y \left(-\frac{2x^2}{r^3} \right) r_x = \frac{2xy}{r^4}$$

putting these value in eqⁿ (i)

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{2xy}{r^3} \frac{\partial \psi}{\partial \theta} + \frac{y^2}{r^4} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{x^2}{r^3} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{2xy}{r^4} \frac{\partial \psi}{\partial \theta} \quad \text{--- (iii)}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{2xy}{r^3} \frac{\partial \psi}{\partial \theta} + \frac{y^2}{r^4} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{x^2}{r^3} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{2xy}{r^4} \frac{\partial \psi}{\partial \theta} \quad \text{--- (iv)}$$

from (iii) and (iv),

$$\frac{\partial^2 \psi}{\partial r^2} = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}$$

This is required solⁿ.

7.a] Find z-transform of $z(a^n)$

$\Rightarrow$ Given $z(a^n)$

We know,

$$z(a^n) = \frac{z}{z-a}$$

††

7.b] Check analyticity of $f(z) = z^2$

$\Rightarrow$ Here, $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$

Equating, $f(z) = u + iv$,
 $u = x^2 - y^2$, $v = 2xy$

$$u_x = 2x \quad v_x = 2y$$

$$u_y = -2y \quad v_y = 2x$$

By C-R eqn.

$u_x = v_y$ and $u_y = -v_x$ hence.

the given func. is analytic

7. C] Solve the partial differential

$$\text{eqn } u_{yy} = u_y$$

$\Rightarrow$ given,

$$u_{yy} = u_y$$

$$\Rightarrow u_{yy} - u_y = 0$$

It's auxiliary eqn is,

$$m^2 - m = 0$$

$$\Rightarrow m = 0, 1$$

$\therefore$ The required soln is,

$$u = A + Be^y$$

#

Set - 5

Year: 2019

Sem: Fall

1.a] State and prove the necessary condition for analyticity. Test the analyticity of the function $f(z) = \log z$.

⇒ Statement: Let $f(z) = u + iv$ be defined and continuous in some (nbd.) of a point $z = x + iy$ and differentiate at $z_0 = x_0 + iy_0$ and z itself. Then at that point $u_x = v_y$ and $u_y = -v_x$.

Proof:

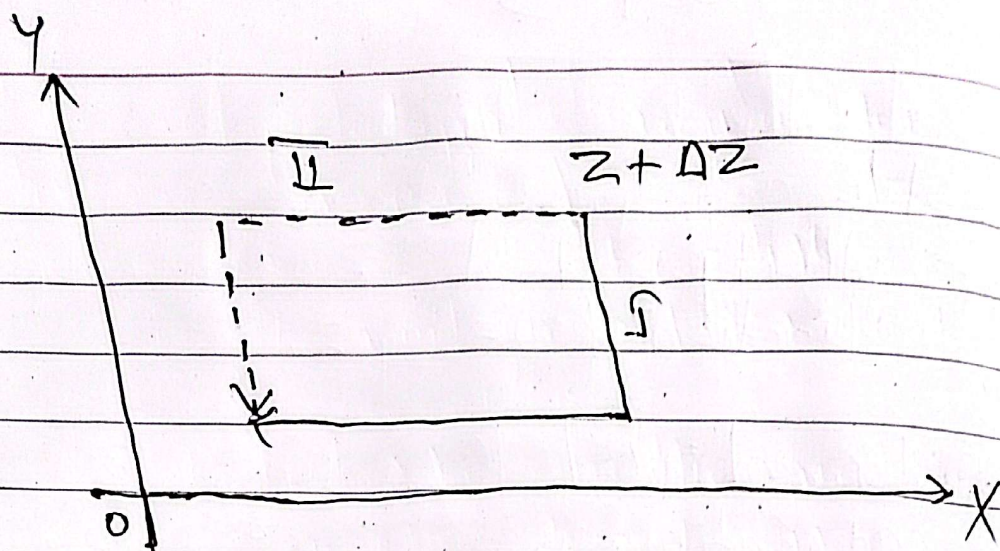
We have. $f(z) = u + iv$ is differentiable then that means $f'(z)$ exists.

$$\text{Where, } f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \left[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) \right] - [u(x, y) + iv(x, y)]$$

$$\frac{(\Delta x + i\Delta y)}{\Delta z + i\Delta y}$$





We have two paths to move on a plane. Starting from z on x -axis and y -axis as you choose.

In eqⁿ (i), Path - (I) where $\Delta y \rightarrow 0$ and $\Delta x \rightarrow 0$. After Δy is zero.
 $\Delta z = \Delta x$ then eqⁿ (i),

$$f'(z) = \lim_{\Delta x \rightarrow 0} \left[\frac{U(x+\Delta x, y) - U(x, y)}{\Delta x} \right] + \lim_{\Delta x \rightarrow 0} i \left[\frac{V(x+\Delta x, y) - V(x, y)}{\Delta x} \right]$$

$$= U_x + iV_x \quad \text{--- (ii)}$$

Path - II where $\Delta x \rightarrow 0$ in eqⁿ (i), After Δx is zero.

$\Delta z = i\Delta y$ then eqⁿ (i) becomes;

$$f'(z) = \lim_{\Delta y \rightarrow 0} \left[\frac{U(x, y+\Delta y) - U(x, y)}{i\Delta y} \right] + \lim_{\Delta y \rightarrow 0} i \left[\frac{V(x, y+\Delta y) - V(x, y)}{i\Delta y} \right]$$

$$= \frac{1}{i} v_y + v_y$$

$$= -i v_y + v_y \quad \text{--- (iii)}$$

from eqn (ii) and (iii),

$$u_x + i v_x = -i v_y + v_y$$

Equating,

$$u_x = v_y \quad \text{and} \quad v_x = -u_y$$

Hence proved.

Here, $f(z) = \log z = \log(re^{i\theta}) = \log r + i\theta$

$f(z) = u + i v$ Comparing, $u = \log r$, $v = \theta$.

$$\frac{\partial u}{\partial x} = \frac{1}{r}, \quad \frac{\partial u}{\partial \theta} = 0$$

$$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial \theta} = 1$$

Also, $\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r} \left(\frac{\partial u}{\partial \theta} \right) = \frac{1}{r} \cdot 1$

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \left(\frac{\partial v}{\partial \theta} \right)$$

Thus the given function is analytic everywhere except $r=0$.

1.6] State Cauchy Integral formula for derivatives. Evaluate $\oint_C \frac{z^6}{(z-1)^6} dz$.

Where C is the unit circle $|z|=1$ counter clockwise.

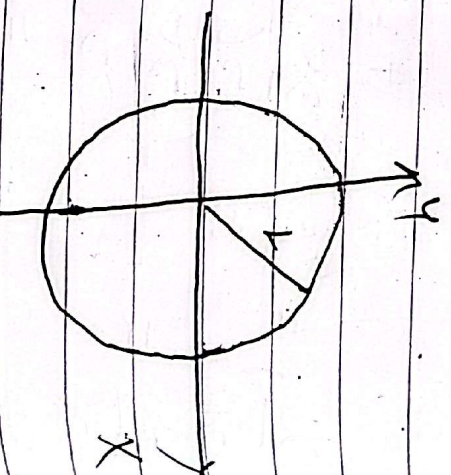
$\Rightarrow$ Statement: Let $f(z)$ be analytic funcⁿ inside and on a simple closed curve C enclosing a simply connected region R and if z_0 is any point on the domain of C . then,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n \text{ is a positive integer.}$$

Here, Given: $\oint \frac{z^6}{(z-1)^6} dz$. $C: |z|=1$.

function is not analytic at $z-1=0$

$\Rightarrow z=1/2$ which is inside the circle.



By Cauchy integral formula, $C: |z|=1$

$$\oint_C \frac{1}{2^6} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{1}{2^6} \frac{2\pi i}{n!} f^{(n)}(z_0) \quad \text{--- (i)}$$

Here, $f(z) = z^6$

Whereas $n=5$,

$$f'(z) = 6z^5$$

$$f''(z) = 30z^4$$

$$f'''(z) = 120z^3$$

$$f^{(4)}(z) = 360z^2$$

$$f^{(5)}(z) = 720z$$

$$f^{(5)}\left(\frac{1}{2}\right) = 360$$

from eqⁿ (i),

$$\oint_C \frac{1}{2^6} \frac{z^6}{(z-1)^6} dz = \frac{1}{2^6} \frac{2\pi i}{5!} \times 360$$

$$= \frac{3\pi i}{32} \quad \#$$

2. a] Integrate $f(z) = \oint_C \frac{e^z + z}{z^3 - z} dz$ around

the unit circle $C: |z| = 1/2$

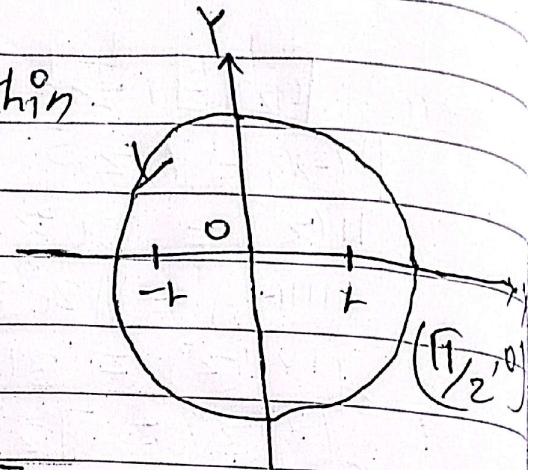
$\Rightarrow$ Given, $f(z) = \frac{e^z + z}{z^3 - z}$

which is not analytic at,

$$z^3 - z = 0 \Rightarrow z = 0, \pm 1$$

these points lie within
C.

Here, residue of $f(z)$ at
 $z = 0$,



$$\begin{aligned} \text{Res.}[f(z), 0] &= \lim_{z \rightarrow 0} z \left[\frac{e^z + z}{z(z^2 - 1)} \right] \\ &= \frac{1}{-1} = -1. \end{aligned}$$

Residue at $z = 1$,

$$\begin{aligned} \text{Res.}[f(z), 1] &= \lim_{z \rightarrow 1} (z-1) \left[\frac{e^z + z}{z(z-1)(z+1)} \right] \\ &= \frac{e^1 + 1}{1 \cdot (2)} = \frac{e+1}{2} \end{aligned}$$

Residue at $z = -1$,

$$\text{Res.}[f(z), -1] = \lim_{z \rightarrow -1} (z+1) \left[\frac{e^z + z}{z(z-1)(z+1)} \right]$$

$$= \frac{e^{-1} + (-1)}{(-1)(-2)} = \frac{e^{-1} - 1}{2}$$

By Cauchy integral formula,

$$\oint_C f(z) dz = 2\pi i [\text{sum of residue}]$$

$$= 2\pi i \left[-1 + \frac{e+1}{2} + \frac{e^{-1}-1}{2} \right]$$

$$= 2\pi i \left[\frac{-2 + e + 1 + e^{-1} - 1}{2} \right]$$

$$= 2\pi i \left[\frac{-2 + e + 1}{2} + \frac{1}{2e} \right]$$

$$\therefore \oint_C \frac{e^z + z}{z^3 - z} dz = 2\pi i \left[-1 + \frac{e}{2} + \frac{1}{2e} \right]$$

#

2.b] Find the fixed points and Normal form of the bilinear transformation $w = \frac{z-1}{z+1}$. Also, determine the nature of this transformation.

$\Rightarrow$ Given,

$$w = \frac{z-1}{z+1}$$

$$\text{Set } w = z,$$

$$z = \frac{z-1}{z+1}$$

$$\Rightarrow z^2 + z = z - 1$$

$$\Rightarrow z^2 = -1$$

$$z = \pm i$$

$\therefore z = \pm i$ is the required fixed point.

By general bilinear transformation,

$$w = \frac{az+b}{cz+d}$$

[Note: cross ratio or nature
are same.]

$$\text{In, } a=1, b=-1 \\ c=1, d=1$$

Normal form, fixed point go to 0 to ∞
gives simple form,
like $w = kz$ or, $w = \frac{1}{z}$.

$$\text{Let } T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\tau = a+d = 1+1 = 2$$

the determinant,

$$\Delta = ad - bc = 1 \cdot 1 - (-1) \cdot 1 = 2$$

Now,

$$\Delta = \tau^2 - 4(ad - bc) = 2^2 - 4(2) = -4$$

Since the discriminant is negative then
the transformation is elliptic #.

[Note: Nature Elliptic $\Delta < 0$, parabolic $\Delta = 0$
Hyperbolic $\Delta > 0$ and Loxodromic $\Delta > 0$
real trace complex trace]

3.a) Define Fourier Integral, choosing a suitable function, show that

$$\int_0^{\infty} \frac{\sin \pi \omega}{\pi^2 - \omega^2} \sin \omega x d\omega = \begin{cases} \frac{1}{2} \sin \pi x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

⇒ Let $f(x)$ be a function of period $2L$ then Fourier integral is,

$$f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$$

Where,

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t dt$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t dt.$$

Here,

$$\text{Let } f(x) = \int_0^{\infty} \frac{\sin \pi \omega \sin \omega x}{\omega} d\omega$$

$$= \begin{cases} \frac{1}{2} \sin \pi x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

by Fourier Sine integral,

$$f(x) = \int_0^{\infty} B(\omega) \sin \omega x \, d\omega \quad (ii)$$

$$\text{Where, } B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \sin \omega t \, dt$$

$$B(\omega) = \frac{2}{\pi} \int_0^{\pi} \frac{\pi}{2} \sin \pi t \sin \omega t \, dt$$

$$= \frac{1}{2} \int_0^{\pi} 2 \sin \pi t \sin \omega t \, dt$$

$$= \frac{1}{2} \left[\int_0^{\pi} [\cos(\pi - \omega)t - \cos(\pi + \omega)t] \, dt \right]$$

$$= \frac{1}{2} \left[\frac{\sin(\pi - \omega)t}{(\pi - \omega)} + \frac{\sin(\pi + \omega)t}{(\pi + \omega)} \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\frac{\sin \omega \pi}{(\pi - \omega)} + \frac{\sin \omega \pi}{(\pi + \omega)} \right]$$

$$= \frac{\sin \omega \pi}{2} \left[\frac{2}{\pi^2 - \omega^2} \right]$$

$$= \frac{\sin \pi \omega}{\pi^2 - \omega^2}$$

Now eqn (ii) becomes,

$$f(x) = \int_0^{\infty} \frac{\sin \pi \omega \sin \omega x}{\pi^2 - \omega^2} d\omega = \text{R.H.S}$$

proved.

3.b] Find the fourier transform of the function $f(x) = e^{-x^2/2}$

⇒ Here, $f(x) = e^{-x^2/2}$

By fourier transform,

$$f(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{i\omega x} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + i\omega x} dx.$$

$$= \int_{-\infty}^{\infty} e^{-\frac{(x+i\omega)^2}{2}} e^{-\frac{\omega^2}{2}} dx$$

$$= e^{-\frac{\omega^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x+i\omega)^2}{2}} dx \quad \text{--- (1)}$$

Put $y = x + i\omega$ then,

$$= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi} \quad \text{then,}$$

eqⁿ (1), becomes,

$$f(\omega) = \sqrt{2\pi} e^{-\frac{\omega^2}{2}} \quad \#$$

4.a] Define Z-transform. State and prove first shifting theorem of Z-transform. Evaluate Z

$\Rightarrow$ Let $f(t)$ be a funcⁿ of time period T where T is non-negative. then, Z-transform of $f(t)$ is,

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

where T is positive number.

Statement: If $Z[f(t)] = F(z)$ then $Z[e^{-at} f(t)] = F[eaTz]$

proof: By definition,

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad \text{--- (i)}$$

$$\begin{aligned} Z[e^{-at} f(t)] &= \sum_{n=0}^{\infty} f(nT) [eaTz]^{-n} \\ &= Z[eaTz] \quad \text{from (i).} \\ &\quad \text{proved.} \end{aligned}$$

here, Given, $\mathcal{Z}(t^2 e^{-bt})$

$$\Rightarrow \mathcal{Z}(t^k) = -\mathcal{Z} \left[\frac{d}{dz} \left(\mathcal{Z} \{ t^{k-1} \} \right) \right]$$
$$\mathcal{Z}(t) = \frac{z}{z-1}$$

and,

$$\mathcal{Z} \{ e^{at} f(t) \} = \left[\mathcal{Z} \{ f(t) \} \right]_{z \rightarrow ze^{-a}}$$

$$\text{Now, } \mathcal{Z}[t^2 e^{-bt}] = \left[\mathcal{Z} \{ t^2 \} \right]_{z \rightarrow ze^b}$$
$$= \left(-\mathcal{Z} \left[\frac{d}{dz} \mathcal{Z}(t) \right] \right)_{z \rightarrow ze^b}$$

$$= \left[-\mathcal{Z} \left[\frac{d}{dz} \left\{ -\mathcal{Z} \left[\frac{d}{dz} \mathcal{Z}(1) \right] \right\} \right] \right]_{z \rightarrow ze^b}$$

$$= \left[-\mathcal{Z} \left[\frac{d}{dz} \left\{ -\mathcal{Z} \left[\frac{d}{dz} \left(\frac{z}{z-1} \right) \right] \right\} \right] \right]_{z \rightarrow ze^b}$$

$$= \left[\mathcal{Z}^2 \frac{d}{dz} \left\{ \frac{z-1}{(z-1)^2} \right\} \right]_{z \rightarrow ze^b}$$

$$= \left[\mathcal{Z}^2 \left(\frac{(z-1)^2(-1) - (-2z)(z-1)}{(z-1)^4} \right) \right]_{z \rightarrow ze^b}$$

$$= T^2 \left[\frac{2(-z+1+2z)}{(z-1)^3} \right] \quad z \rightarrow ze^{bT}$$

$$= T^2 \left[\frac{z^2+z}{(z-1)^3} \right] \quad z \rightarrow ze^{bT}$$

$$= T^2 \left[\frac{ze^{2bT} + ze^{bT}}{(ze^{bT}-1)^3} \right] \quad \#.$$

or,

$$= T^2 e^{-bT} \left[\frac{2(z+e^{-bT})}{(z-e^{-bT})^3} \right]$$

#

4.6] Solve the differential eqⁿ by using z-transform.

$$y_{n+2} - 3y_{n+1} + 2y_n = 4^n, \quad y_0 = 0, y_1 = 1$$

⇒ Given,

$$y_{n+2} - 3y_{n+1} + 2y_n = 4^n, \quad y_0 = 0, y_1 = 1$$

①

Taking z -transform on both side,

$$z(y_{n+2}) - 3z(y_{n+1}) + 2z(y_n) = z(4^n)$$

$$\Rightarrow z^2(\bar{y} - y_0 - \frac{y_1}{z}) - 3z(\bar{y} - y_0) + 2\bar{y} = \frac{z}{(z-4)}$$

$$\bar{y}(z^2 - 3z + 2) - z = \frac{z}{(z-4)}$$

$$\bar{y} = \frac{z^2 - 3z}{(z-4)(z-2)(z-1)}$$

Taking z^{-1} on both sides,

$$y_n = z^{-1} \left(\frac{z^2 - 3z}{(z-1)(z-2)(z-4)} \right) \text{--- (1)}$$

$$z^{n-1} f(z) = \frac{z^{n+1} - 3z^n}{(z-1)(z-2)(z-4)}$$

poles at, $z = 1, 2, 4$

which is all are simple pole.

Resi. at $z=1$,

$$\text{Res.}[f(z), 1] = \lim_{z \rightarrow 1} \left[\cancel{(z-1)} \frac{(z^{n+1} - 3z^n)}{(\cancel{z-1})(z-2)(z-4)} \right]$$

$$= \frac{1-3}{(-1)(-3)} = \frac{-2}{3}$$

Residue at $z=2$,

$$\text{Res.}[f(z), 2] = \lim_{z \rightarrow 2} \left[\cancel{(z-2)} \frac{(z^{n+1} - 3z^n)}{(z-1)\cancel{(z-2)}(z-4)} \right]$$

$$= \frac{2^{n+1} - 3 \cdot 2^n}{(1)(-2)}$$

$$= 2^n \left(\frac{2-3}{-2} \right) = \frac{2^n}{-2} = 2^{n-1}$$

Residue at $z=4$,

$$\text{Res.}[f(z), 4] = \lim_{z \rightarrow 4} \left[\cancel{(z-4)} \frac{(z^{n+1} - 3z^n)}{(z-1)(z-2)\cancel{(z-4)}} \right]$$

$$= \frac{4^{n+1} - 3 \cdot 4^n}{(3)(2)}$$

$$= \frac{4^n(4-3)}{6} = \frac{4^n}{6}$$

Now, eqⁿ (i) becomes,

$$y_n = -\frac{2}{3} + 2^{n-1} + \frac{4^n}{6}$$

#

5. a] Derive one dime. wave equation with solution.

⇒ We know, one dimen. wave eqⁿ,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (1)}$$

$$\text{Let } y(x,t) = f(x) \cdot G(t) \quad \text{--- (2)}$$

Solution of one dime. wave eqⁿ.

Differentiating eqn (ii) w.r. to x and t ,

$$\frac{\partial y}{\partial x} = f' G \Rightarrow \frac{\partial^2 y}{\partial x^2} = f'' G$$

$$\frac{\partial y}{\partial t} = f \ddot{G} \Rightarrow \frac{\partial^2 y}{\partial t^2} = f \ddot{G}$$

eqn (i) becomes,

$$\ddot{F} \ddot{G} = c^2 f'' G$$
$$\Rightarrow \frac{\ddot{G}}{c^2 G} = \frac{f''}{f} = k \text{ (say)} \quad \text{This gives}$$

$$\ddot{G} - c^2 k G = 0 \quad \text{--- (iii)}$$

$$f'' - kf = 0 \quad \text{--- (iv) which ordinary diff. eqn.}$$

Solving them,

$$F = A e^{kx} + B e^{-kx}$$

Where k is positive

$$\text{let } \lambda^2 = k \text{ then, } k = \pm \lambda^2$$

$$F = A e^{\lambda x} + B e^{-\lambda x}$$

$$\text{if } k = -p^2 \text{ then where } k \text{ is (-ve)}$$

$$F'' + P^2 F = 0$$

$$m^2 + p^2 = 0$$

$$m = \pm pi \text{ Then,}$$

$$f(x) = A \cos px + B \sin px$$

$$\text{we have, } f(0) = A = 0$$

$$f(L) = B \sin pL = 0 \quad B \neq 0$$

$$\sin pL = 0$$

$$pL = n\pi$$

$$p = \frac{n\pi}{L} \text{ for } n \text{ is integer.}$$

$$B \neq 0 ; \text{ setting } B = r,$$

$$f(x) = \sin\left(\frac{n\pi}{L}\right)x, \text{ we obtain infinitely many sol}^n.$$

$$f(x) = \sum_n f_n(x) \text{ thus,}$$

$$f_n(x) = \sin \frac{n\pi x}{L} \text{ for } n = 1, 2, 3, \dots \quad \text{--- (X)}$$

$$\text{then, } k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$$

$$\text{solving (iii), } G'' - G^2 k G = 0$$

$$\ddot{G} + \lambda_n^2 G = 0 \quad \text{when } \lambda_n = \frac{C_n \pi}{L}$$

$$m^2 + \lambda_n^2 = 0 \Rightarrow m = \pm \lambda_n i$$

It's soln:

$$G(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t$$

The required solution of eqn (1),

$$y(x, t) = f(x) \cdot G(t)$$

$$y_n(x, t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi x}{L}$$

— (A)

Which is required soln.

5.6] Find the temp $y(x, t)$ which is distributed laterally in a insulated copper bar ($c^2 = 1.158 \text{ cm}^2/\text{sec}$), 100 cm long and of constant cross section whose end points at $x=0$ and $x=100$ are kept at 0°C and its initial temp is $f(x) = \sin^3(0.01)\pi x$

$\Rightarrow$ Given, $C^2 = 1.158 \text{ cm}^2/\text{sec}$, $L = 100 \text{ cm}$
 $x=0$ and $x=100$; i.e. $\lambda_n = \frac{n\pi}{L}$

Condⁿ, $y(0, t) = 0 = y(100, t)$

initial deflection $f(x) = \sin^3(0.01)\pi x$.
 $= y(x, 0)$ — (i)

The required temp^l is given by -

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \cdot e^{-\lambda_n^2 t} \quad \text{(ii)}$$

Where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Put $L=100$,

$$y(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \quad \text{(iii)}$$

from eqⁿ (i) and eqⁿ (iii),

$$\sin^3(0.01)\pi x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \quad \text{(iv)}$$

$$\left[\because \sin 3A = 3\sin A - 4\sin^3 A \right]$$

$$\sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

So, (iV) becomes -

$$\sin^3(0.01)\pi x = \frac{3\sin(0.01)\pi x - \sin(0.03)\pi x}{4}$$

$$\frac{1}{4} (3\sin(0.01)\pi x - \sin(0.03)\pi x)$$

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

$$= B_1 \sin \frac{\pi x}{100} + B_2 \sin \frac{2\pi x}{100} + B_3 \sin \frac{3\pi x}{100} + \dots$$

$$= B_1 \sin(0.01)\pi x + B_2 \sin(0.02)\pi x + B_3 \sin(0.03)\pi x + \dots$$

Equating the coefficient

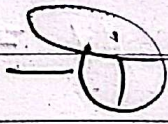
$$B_1 = 3/4, B_3 = -1/4, B_2 = 0 = B_4 = B_5 = \dots$$

Now, eqn (i) becomes,

$$\begin{aligned}
 y(x, t) &= \frac{3}{4} \sin(0.01) \pi x \cdot e^{-\left(\frac{1.158 \pi}{100}\right)^2 t} - \frac{1}{4} \sin(0.03) \pi x \cdot e^{-\left(\frac{1.158 \times 21 \pi}{100}\right)^2 t} \\
 &= \frac{3}{4} \sin(0.01) \pi x \cdot e^{-0.001158 \pi^2 t} - \frac{1}{4} \sin(0.03) \pi x \cdot e^{-0.01029 t}
 \end{aligned}$$

#

6.a] Express the Laplacian $\nabla^2 y = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$ in Polar Coordinate.

⇒ The Laplacian Cartesian coordinate is,
 $\nabla^2 y = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$ 

By Polar Coordinate to change,
 $x = r \cos \theta$, $y = r \sin \theta$.

Then, $u_x = \frac{\partial u}{\partial x} = u_r r_x + u_\theta \theta_x$

Again,

$$u_{xx} = (u_r r_x)_x + (u_\theta \theta_x)_x$$

$$= (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x \theta_x + u_\theta \theta_{xx}$$

$$= (u_{rr} r_x + u_{r\theta} \theta_x) r_x + u_r r_{xx} +$$

$$(u_{\theta r} r_x + u_{\theta\theta} \theta_x) \theta_x + u_\theta \theta_{xx}$$

We have,

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}$$

and,

$$\theta_x = \frac{1}{1 + (y/x)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{r^2}$$

Again, $r_{xx} = \frac{r \cdot x_{rx}}{r^2} = \frac{1}{r} - \frac{x^3}{r^3} = \frac{y^3}{r^3}$

and

$$\theta_{xx} = -y \left(\frac{-2}{r^3} \right) r_x = \frac{2xy}{r^4}$$

Putting in eqn (i).

$$u_{xx} = \frac{x^2}{r^2} u_{rr} - \frac{2xy}{r^3} u_{r\theta} + \frac{y^2}{r^4} u_{\theta\theta} + \frac{y^2}{r^3} u_r + \frac{2xy}{r^4} u_\theta$$

— (iii)

$$u_{yy} = \frac{y^2}{r^2} u_{rr} + \frac{2xy}{r^3} u_{r\theta} + \frac{x^2}{r^4} u_{\theta\theta} + \frac{y^2}{r^3} u_r - \frac{2xy}{r^4} u_\theta$$

— (iv)

from (iii) and (iv),

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

Which is required laplacian u in polar coordinate.

6.6] Define partial differential equation with suitable example. By separating the variable solve $y_{xx} + y_{yy} = 0$.

→ A eqn involving one or more partial derivatives of an (unknown) function of two or more independent variable is called partial differential eqn.

for example: one dimen. wave eqn: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

one-D-heat eqn: $\frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2}$
etc.

Here, Given:

$$y_{xx} + y_{yy} = 0$$

Let the general solution be,

$$y(x, y) = F(x) \cdot G(y) \quad \text{--- (i)}$$

We have, two dimension. Laplace eqn.

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0 \quad \text{--- (ii)}$$

Substitute,

$$\frac{d^2 F G}{dx^2} + G \frac{d^2 F}{dy^2} = 0$$

$$\frac{1}{F} \frac{d^2 F}{dx^2} = - \frac{1}{G} \frac{d^2 G}{dy^2} = \lambda \text{ (say) then}$$

$$\frac{d^2 F}{dx^2} = \lambda F \quad \text{--- } (*)$$

$$\frac{d^2 G}{dy^2} = -\lambda G \quad \text{--- } (**)$$

Case-I let $\lambda = k^2 > 0$

$$F(x) = A \cosh(kx) + B \sinh(kx)$$

$$G(y) = C \cos(ky) + D \sin(ky)$$

Case-II $\lambda = -k^2 < 0$

$$F(x) = A \cos(kx) + B \sin(kx)$$

$$G(y) = C \cosh(ky) + D \sinh(ky) \quad \text{then,}$$

eqn $(*)$ becomes,

$$\psi(x, y) = (A \cos(kx) + B \sin(kx)) \cdot (C \cosh(ky) + D \sinh(ky))$$

or similar forms

7. a] Express $f(z) = \sinh z$ in the terms of $u + iv$.

$\Rightarrow$ Here, $f(z) = \sinh z$

$$= e^z - e^{-z}$$

$$= \frac{1}{2} (e^{x+iy} - e^{-(x+iy)})$$

$$= \frac{1}{2} [e^x e^{iy} - e^{-x} e^{-iy}]$$

$$= \frac{1}{2} [e^x (\cos y + i \sin y) - e^{-x} (\cos y - i \sin y)]$$

$$= \frac{1}{2} [e^x \cos y + i e^x \sin y - e^{-x} \cos y + e^{-x} i \sin y]$$

$$= \frac{1}{2} [(e^x \cos y - e^{-x} \cos y) + i (e^x \sin y + e^{-x} \sin y)]$$

$$= \frac{1}{2} [\cos y (e^x - e^{-x}) + i (e^x + e^{-x}) \sin y]$$

$$u + iv = \cos y \sinh x + i \cosh x \sin y$$

Equating

$$u = \cos y \sinh x$$

$$v = \cosh x \sin y$$

7.6] Solve $y_{xx} - y = 0$ by using separation of variable.

⇒ Given $y_{xx} - y = 0$

$$\frac{d^2 y}{dx^2} - y = 0$$

it's auxiliary eqⁿ is,

$$m^2 - 1 = 0$$

$$m = \pm 1$$

then its solution is,

$$y = A e^x + B e^{-x}$$

#

7.7] find z-transform of $(n 4^n)$

⇒ we know,

$$Z\{n a^n\} = \frac{a \cdot z}{(z-a)^2} \quad |z| > |a|$$

then

$$Z[n 4^n] = \frac{4z}{(z-4)^2}$$

#

~~500~~
~~500~~