

Complex Analysis

Complex number: $z = x + iy = (x, y)$

eg. $z = 3 + i4 = (3, 4)$

No-1e:

② $i \cdot i = -1$, $\sqrt{-1} = i$

Conjugate/Complex conjugate

$$z = x + iy \Rightarrow \bar{z} = x - iy$$

$$\sqrt{-1} = i$$

Note: a) $Z = Z^H$

$$b) \quad \frac{z + \bar{z}}{2} = \operatorname{Re}(z)$$

$$\left[\begin{array}{l} z = x + iy \\ \operatorname{Re}(z) = x \\ \operatorname{Im}(z) = y \end{array} \right]$$

magnitude of a complex number $z = x + iy$

$$|z| = \sqrt{x^2 + y^2}$$

Triangle Law. (a)

Note: $|x+y| \leq |x| + |y|$ $\left| \frac{x}{y} \right| = \frac{|x|}{|y|}$

$$\left| \frac{x}{4} \right| = \frac{|x|}{|4|}$$

$$z = re^{i\theta} = x + iy$$

Polar Co-ordinate:

$$x = r \cos \theta, y = r \sin \theta, r^2 = x^2 + y^2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now,

$$z = x + iy = r \cos \theta + i r \sin \theta \\ = r (\cos \theta + i \sin \theta)$$

Q. $z = 1 + i$ change into polar.

$$\Rightarrow \text{Comparing } z = x + iy \\ x = 1, y = 1$$

$$z = r (\cos \theta + i \sin \theta)$$

Now,

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

So,

$$z = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

De-Moivre's theorem.

$$[\text{Note: } (r(\cos \theta + i \sin \theta))^n = r^n (\cos n\theta + i \sin n\theta)]$$

Operation on complex number:

let $z = a + ib$, $w = c + id$ then find

$$z + w, z - w, z \times w, \frac{z}{w}$$

$$\Rightarrow z + w = a + ib + c + id = a + c + i(b + d)$$

$$\text{Ex: } z = 2 + 3i, w = 4 + i \Rightarrow z + w = 6 + 4i$$

$$\Rightarrow z - w = 2 + 3i - 4 - i = -2 + 2i$$

$$\Rightarrow z \cdot w = (a + ib) \cdot (c + id) \\ = ac + iad + ibc - bd \quad [i^2 = -1] \\ = ac - bd + i(ad + bc) \\ = (ac - bd, ad + bc)$$

$$\Rightarrow \frac{z}{w} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id} \\ = \frac{ac - iad + ibc + bd}{c^2 + d^2} \quad [i^2 = -1]$$

$$= \frac{ac + bd - i(ad - bc)}{c^2 + d^2}$$

$$= \left(\frac{ac + bd}{c^2 + d^2}, \frac{ad - bc}{c^2 + d^2} \right)$$

Q. $f(z) = z^2 + 1$.

Let $z = x + iy$

$$f(z) = (x + iy)^2 + 1$$

$$= x^2 + 2xyi - y^2 + 1$$

$$= (x^2 - y^2 + 1) + 2xyi$$

Now,

$$\operatorname{Re}\{f(z)\} = x^2 - y^2 + 1$$

$$\operatorname{Im}\{f(z)\} = 2xy$$

Q. Find the value of $\operatorname{Re} f$ and $\operatorname{Im} f$ at given point:

$$f(z) = \frac{z-2}{z+2} \text{ at } z = 4i$$

Let $z = x + iy$

then,

$$f(z) = \frac{x + iy - 2}{x + iy + 2}$$

$$= \frac{(x + iy - 2)(x - iy + 2)}{(x + iy + 2)(x - iy + 2)}$$

$$= \frac{x^2 - iyx + 2x + ixy - y^2 + 4}{x^2 - iyx + 2x + ixy - y^2 + 4}$$

$$f(z) = \frac{4i - 2}{4i + 2} \times \frac{4i - 2}{-4i + 2} = \frac{3}{5} - \frac{4}{5}i$$

$$= \frac{-16 - 8i - 8i + 4}{(4)^2 - (2)^2}$$

$$= \frac{-12 - 16i}{-16 + 4} = \frac{-12 - 16i}{-12}$$

$$= 1 + \frac{4}{3}i \quad \operatorname{Re} f = \frac{3}{5}, \operatorname{Im} f = \frac{4}{5}$$

Note: $e^{i\theta} = \cos\theta + i\sin\theta$
 $e^{-i\theta} = \cos\theta - i\sin\theta$

Express the function $\cosh z$ in the form of $f(z) = u + iv$

$\Rightarrow$ Since, $\left[\cosh x = \frac{e^x + e^{-x}}{2} \right], \left[\sinh x = \frac{e^x - e^{-x}}{2} \right]$

$$\begin{aligned} f(z) = \cosh z &= \frac{e^z + e^{-z}}{2} \\ &= \frac{1}{2} \left[e^{x+iy} + e^{-(x+iy)} \right] \\ &= \frac{1}{2} \left[e^x \cdot e^{iy} + e^{-x} \cdot e^{-iy} \right] \\ &= \frac{1}{2} \left[e^x (\cos y + i \sin y) + e^{-x} (\cos y - i \sin y) \right] \end{aligned}$$

part 2

$$\begin{aligned} &= \frac{1}{2} \left[e^x \cos y + i e^x \sin y + e^{-x} \cos y - i e^{-x} \sin y \right] \\ &= \frac{1}{2} \left[(e^x \cos y + e^{-x} \cos y) + i (e^x \sin y - e^{-x} \sin y) \right] \\ &= \frac{1}{2} \left[\cos y (e^x + e^{-x}) + i \sin y (e^x - e^{-x}) \right] \\ &= \cos y \cosh x + i \sin y \sinh x \end{aligned}$$

$\therefore u = \cos y \cosh x, v = \sin y \sinh x$

Limit of a function of a Complex Variable.

Let $f(z)$ be a single valued function of complex variable z , then $f(z)$ has limit $l = \alpha + i\beta$ as z tends to z_0 if for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$|f(z) - l| < \epsilon, \text{ where } |z - z_0| < \delta$$

i.e. $\lim_{z \rightarrow z_0} f(z) = l$

Continuity of a function of a Complex Variable.

Let $f(z)$ be a complex valued function. It is said to be continuous at z_0 for every $\epsilon > 0$ there exists $\delta > 0$ such that,

$$|f(z) - f(z_0)| < \epsilon \text{ Whenever } |z - z_0| < \delta$$

i.e. $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

③ Epsolune

for the continuity of a function.

- ① $f(z)$ must defined
- ② $f(z)$ must exists.
- ③ above two value must equal.

Q. Test the continuity of function.

$$f(z) = \frac{x^2 y}{x^4 + y^2} \quad \text{for } z \neq 0$$
$$= 0 \quad \text{for } z = 0$$

⇒ Now, we have show that, the function is continuous / discon.

consider that, the $y = mx^2$ along which x and y vary and comes to close to the origin.

choose $x = a$, where a is very small and close to zero.

As $z \rightarrow 0 \Rightarrow a \rightarrow 0$

then,

$$z = x + iy = a + imx^2 = a + ima^2$$

$$f(z) = \frac{x^2 y}{x^4 + y^2} = \frac{a^4 m}{a^4 + a^4 m^2}$$

$$\Rightarrow f(z) = \frac{m}{1+m^2}$$

Put $z \rightarrow 0$ on both side

$$\lim_{z \rightarrow 0} f(z) = \lim_{a \rightarrow 0} \left(\frac{m}{1+m^2} \right) = \frac{m}{1+m^2}$$

functional value of $f(z) = 0$ so. The given function $f(z)$ is discontinuous.

Q. show that the function $f(z)$ defined by:

$$f(z) = \left[\frac{2x^2y}{x^2+y^2} \right] \text{ for } z \neq 0$$

$$= 0 \text{ for } z=0 \text{ is conti. at } z=0$$

We have $f(0)$ is defined; $f(0)=0$

We have show

$$\lim_{z \rightarrow 0} f(z) \text{ exists.}$$

$\Rightarrow$ Here put $x=a$, $y=am$ when $x \rightarrow 0 \Rightarrow$

$$a \rightarrow 0, y \rightarrow 0 \Rightarrow a \rightarrow 0$$

$$\lim_{z \rightarrow 0} f(z) = \lim_{z \rightarrow 0} \left[\frac{2x^2y}{x^2+y^2} \right]$$

$$= \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left[\frac{2x^2y}{x^2+y^2} \right]$$

$$= \lim_{a \rightarrow 0} \left[\frac{2a^3m}{a^2+a^2m^2} \right]$$

$$= \lim_{a \rightarrow 0} \left[\frac{2am}{1+m^2} \right]$$

$$= 0 \text{ exist.}$$

Hence,

$\therefore$ functional value of $f(z)=0$. So the given function is continuous at $z=0$

Differentiability of a complex number.

Let $f(z)$ be the complex valued function, then $f(z)$ is said to be differentiable at z_0 if the following limit exists.

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

or,

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Q. Prove that $f(z) = z^2$ is differentiable.

⇒ We have.

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{z^2 + 2z\Delta z + \Delta z^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} (2z + \Delta z)$$

$$= 2z$$

Q. Is the function $f(z) = \bar{z}$ is differentiable.

⇒ We have.

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$f'(z) = \begin{cases} 1 & \text{if } \Delta y \rightarrow 0 \\ -i & \text{if } \Delta x \rightarrow 0 \end{cases}$$

which is oscillatory. Thus, $f'(z)$ is not fixed implies $f(z)$ is not differentiable.

function differentiation:
 $\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$

or,
 $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$

eg. find the derivatives of $f(z) = (z - 4i)^8$
 at $z_0 = 5 + 4i$

⇒ Since, $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ if the point z_0 is given,

$$= \lim_{z \rightarrow (5+4i)} \frac{(z-4i)^8 - (5+4i-4i)^8}{z - (5+4i)} \left[\frac{0}{0} \right]$$

By L-hospital's Rule,

$$= \lim_{z \rightarrow (5+4i)} \frac{8(z-4i)^7 - 0}{1-0}$$

$$= 8(5+4i-4i)^7$$

$$= 8 \times 5^7$$

#

Q. find the derivatives of $f(z) = \frac{z-i}{z+i}$ at $z_0 = i$

⇒ Since, $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$$= \lim_{z \rightarrow i} \frac{\frac{z-i}{z+i} - \frac{z_0-i}{z_0+i}}{z - z_0}$$

$$= \lim_{z \rightarrow i} \frac{z-i}{(z+i)(z/i)} \quad [z_0 = i]$$

$$= \lim_{z \rightarrow i} \frac{1}{z+i}$$

$$= \frac{1}{2i}$$

$$= \frac{1}{2i} \times \frac{i}{i}$$

$$= -\frac{i}{2} \quad [\because i^2 = -1]$$

V.V.S Long Question -

Cauchy Riemann's Equation and Laplace Equation:

⇒ Cauchy Riemann's Equation:

Definition: Analytic function $f(z)$,

A funcⁿ $f(z)$ is said to be analytic in a domain D if and only if the first partial derivatives of u and v satisfies the two, so-called Cauchy Riemann's equation.

$$u_x = v_y, \quad v_x = -u_y \text{ every where in } D.$$

eg. show that $f(z) = z^2 + z$ is analytic in the domain D by using C-R eqⁿ:

⇒ Now, let $z = x + iy$ then,

$$\begin{aligned} f(z) &= (x + iy)^2 + x + iy \\ &= x^2 + 2xyi + (-y^2) + x + iy \\ &= x^2 + x - y^2 + i(2xy + y) \end{aligned}$$

$$u = x^2 - y^2 + x$$

$$v = 2xy + y$$

Now. C-R eqⁿ:

$$\begin{aligned} u_x &= 2x + 1 & v_x &= 2x \\ v_y &= 2x + 1 & u_y &= -2y \end{aligned}$$

So, $u_x = v_y$ and $v_x = -u_y$ Hence proved.

I.m.f

Analytic function / Holomorphic function / Regular function.

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbouring points of that point.

Ex. $f(z) = \frac{1}{z}$ which is not analytic at $z=0$.

2] Entire function: A funcⁿ $f(z)$ is said to be entire funcⁿ which is analytic every where in finite plane.
 $f(z) = e^z$ is analytic every where.

C-F eqn:

Necessary Condition of C-F eqn:

The necessary condition for Complex function $f(z) = u(x, y) + i v(x, y)$ to be analytic in region R .

if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore u_x = v_y$ & $v_x = -u_y$

Q. Prove that, $f(z) = e^z$ is analytic by using C-F eqn.

$\Rightarrow f(z) = e^z$ — (i)

$f(z) = e^{x+iy} = e^x \cdot e^{iy}$

$= e^x (\cos y + i \sin y)$

$= e^x \cos y + i e^x \sin y$

Hence, $u = e^x \cos y$
 $v = e^x \sin y$

Now. By C-F eqn: -

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ — (ii)

$\Rightarrow \frac{\partial e^x \cos y}{\partial x} = \frac{\partial e^x \sin y}{\partial y}$

$\Rightarrow e^x \cos y = e^x \cos y$

and,

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ — (iii)

$\Rightarrow \frac{\partial e^x \sin y}{\partial x} = -\frac{\partial e^x \cos y}{\partial y}$

$\Rightarrow e^x \sin y = -e^x \sin y$

Hence, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ then

the given function is analytic.

C.W.

8. is the funcⁿ $zxy + i(x^2 - y^2)$ is analytic?

⇒ here.

$$U = zxy$$

$$V = x^2 - y^2$$

then. $U_x = V_y$

$$\frac{\partial (zxy)}{\partial x} = \frac{\partial (x^2 - y^2)}{\partial y}$$

$$\Rightarrow zy = -2y$$

and.

$$U_y = 2x$$

$$V_x = 2x$$

$$\therefore U_x \neq V_y \text{ and } U_y \neq -V_x.$$

then, it is not analytic.

$$Z = r[\cos\theta + i\sin\theta]^n$$

$$= r^n [\cos n\theta + i\sin n\theta] \quad (e^{i\theta} = \cos\theta + i\sin\theta)$$

Note: Polar form of Cauchy-Riemann condition.

$$\frac{\partial U}{\partial r} = \frac{1}{r} \frac{\partial V}{\partial \theta} \quad \text{--- (i)}$$

$$\frac{\partial U}{\partial \theta} = -r \frac{\partial V}{\partial r} \quad \text{--- (ii)}$$

(A)

Example: show that the funcⁿ $f(z) = z^n$ is analytic.

$$\Rightarrow \text{here } Z = [r(\cos\theta + i\sin\theta)]^n$$

$$= r^n [\cos n\theta + i\sin n\theta].$$

$$= r^n \cos n\theta + i r^n \sin n\theta.$$

Hence, $U = r^n \cos n\theta$, $V = r^n \sin n\theta$.

$$\frac{\partial U}{\partial r} = \cos n\theta \frac{\partial r^n}{\partial r} = n \cos n\theta r^{n-1}$$

$$\frac{1}{r} \frac{\partial V}{\partial \theta} = \frac{1}{r} \frac{\partial [r^n \sin n\theta]}{\partial \theta} = r^{n-1} \cdot n \cos n\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial [r^n \cos n\theta]}{\partial \theta} = r^n (-n \sin n\theta) \\ = -r^n n \sin n\theta.$$

$$r \frac{\partial v}{\partial r} = r \frac{\partial [r^n \sin n\theta]}{\partial r} = r \cdot \sin n\theta \cdot n \cdot r^{n-1} \\ = n \cdot r^n \sin n\theta.$$

$$\text{Hence, } \frac{\partial v}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

and

$$\frac{\partial v}{\partial \theta} = -r \frac{\partial v}{\partial r} \quad \text{Hence, the}$$

funcⁿ is analytic.

P.V 5-marks

Q. prove that $\log z$ is analytic everywhere except at $(0,0)$.

⇒ Here $z = \log z$

$$= \log (r \cdot e^{i\theta})$$

$$= \log r + \log e^{i\theta}.$$

$$= \log r + i\theta \log e \quad [\because \log e = 1] \\ = \log r + i\theta$$

Here,

$$U = \log r \quad \text{and} \quad V = \theta.$$

$$\text{Now, } \frac{\partial v}{\partial r} = \frac{\partial \log r}{\partial r} = \frac{1}{r}$$

$$\frac{1}{r} \frac{\partial v}{\partial \theta} = \frac{1}{r} \frac{\partial \theta}{\partial \theta} = \frac{1}{r}$$

$$\text{and, } \frac{\partial v}{\partial \theta} = \frac{\partial \log r}{\partial \theta} = 0$$

$$r \frac{\partial v}{\partial r} = r \frac{\partial \theta}{\partial r} = 0$$

$$\text{Hence, } \frac{\partial v}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and}$$

$$\frac{\partial v}{\partial \theta} = r \frac{\partial v}{\partial r} \quad \text{So, the funcⁿ is} \\ \text{analytic.}$$

Laplace Equation and harmonic funcⁿ.

The equation of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is known as two dimension Laplace ~~trans form~~ Equation.

And $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$ is known as three dimen. Laplace eqⁿ.

At ~~last~~ ~~off~~: $\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$ is the polar form of Laplace eqⁿ.

A real funcⁿ of two variable x and y that posses continuous second order partial derivatives and that satisfies Laplace eqⁿ is called harmonic funcⁿ.

Conjugate funcⁿ: If u and v are harmonic funcⁿ such that $u + iv$ is analytic then each is called harmonic conjugate funcⁿ of each other.

Q. proof that u and v are harmonic funcⁿs if $f(z) = e^z$.

$$\Rightarrow f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x [\cos y + i \sin y] = e^x \cos y + i e^x \sin y$$

$$\Rightarrow V(x, y) = e^x \cos y \quad \text{--- (i)}$$

$$U(x, y) = e^x \sin y \quad \text{--- (ii)}$$

for U harmonic, we have to show:

$$U_{xx} + U_{yy} = 0$$

$$\Rightarrow \frac{\partial}{\partial x} \frac{\partial}{\partial x} e^x \cos y + \frac{\partial}{\partial y} \frac{\partial}{\partial y} e^x \cos y$$

$$= \frac{\partial}{\partial x} e^x \cos y + \frac{\partial}{\partial y} (-e^x \sin y)$$

$$= e^x \cos y - e^x \cos y$$

$$= 0 = \text{R.H.S proved.}$$

For V harmonic, we have to show:

$$V_{xx} + V_{yy} = 0$$

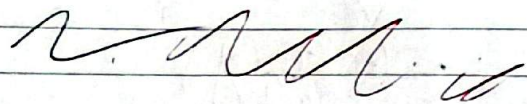
$$\Rightarrow \frac{\partial}{\partial x} \frac{\partial}{\partial x} e^x \sin y + \frac{\partial}{\partial y} \frac{\partial}{\partial y} e^x \sin y$$

$$= \frac{\partial}{\partial x} e^x \sin y + \frac{\partial}{\partial y} e^x \cos y$$

$$= e^x \sin y - e^x \sin y$$

$$= 0 = \text{RHS.}$$

So, the funcⁿ is V and V are harmonic funcⁿ.



Construction of harmonic conjugate:
Milne Thomson Method.

① To find $f(z)$: If V is given,
then $f(z) = \int \psi_1(z, 0) dz - i \int \psi_2(z, 0) dz + c$

$$\psi_1(z, 0) = \frac{\partial V(x, y)}{\partial x}$$

$$\psi_2(z, 0) = \frac{\partial V(x, y)}{\partial y} \quad \text{Where } c \text{ is integrating constant.}$$

① If V is given,

$$f(z) = \int \psi_1(z, 0) dz + i \int \psi_2(z, 0) dz + c$$

$$\psi_1(z, 0) = \frac{\partial V(x, y)}{\partial x}$$

P.V $\psi_2(z, 0) = \frac{\partial V(x, y)}{\partial y}$

for example: Find the analytic funcⁿ $f(z)$.
whose real part is $e^x(x \cos y - y \sin y)$.

$\Rightarrow$ Since, $V(x, y) = e^x(x \cos y - y \sin y)$

We have if $V(x, y)$ is given.

$$f(z) = \int \psi_1(z, 0) dz - i \int \psi_2(z, 0) dz + c$$

Where.

$$\psi_1(x, y) = \frac{\partial U}{\partial x} \text{ and } \psi_2(x, y) = \frac{\partial U}{\partial y}$$

$$= \frac{\partial (xe^x \cos y - e^x y \sin y)}{\partial x}$$

$$= \cos y [x \cdot e^x + e^x] - e^x y \sin y$$

$$= xe^x \cos y + e^x \cos y - e^x y \sin y$$

$$= ze^z \cos 0 + e^z \cos 0 - e^z \cdot 0 \sin 0$$

$$= ze^z + e^z$$

and.

$$\psi_2(x, y) = \frac{\partial U}{\partial y} = \frac{\partial (e^x x \cos y - e^x y \sin y)}{\partial y}$$

$$= x \cdot e^x (-\sin y) - e^x [y \cdot \cos y + \sin y \cdot 1]$$

$$= -xe^x \sin y - e^x y \cos y - e^x \sin y$$

$$= -ze^z \sin 0 - e^z \cdot 0 \cdot \cos 0 - e^z \sin 0$$

$$= 0$$

$$\text{Now. } f(z) = \int (ze^z + e^z) dz - i \int 0 dz + c$$

$$= \int ze^z dz + \int e^z dz$$

$$= ze^z - \int \left[\frac{dz}{dz} \int e^z dz \right] + e^z$$

$$= ze^z - e^z + e^z$$

$$= ze^z + c$$

∴

∴ V find Gen.

$$f(z) = (x+iy) e^{(x+iy)}$$

=

we find V.

Complex function

↳ Harmonic of U } $f(z) = U + iV$
 ↳ Harmonic of V }

if U is given: $f(z) = \int \phi_1(z,0) dz - i \int \phi_2(z,0) dz + C$

if V is given: $f(z) = \int \phi_1(z,0) dz + i \int \phi_2(z,0) dz + C$

Q. State and prove that necessary condition for analyticity of C-R equation.

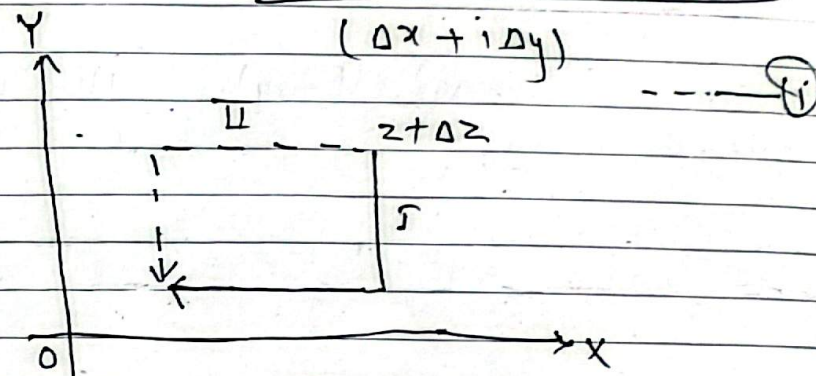
⇒ Statement: Let $f(z) = U + iV$ be defined and continuous in some (nbd.) of a point $z = x + iy$ and differentiable at $z_0 = x_0 + iy_0$ and z itself. Then at that point $U_x = V_y$ and $U_y = -V_x$.

Proof: We have $f(z) = U + iV$ is differentiable then that means $f'(z)$ exists.

Where, $f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$

$$f_x = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad f_y = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}$$

$$= \lim_{\Delta z \rightarrow 0} [U(x + \Delta x, y + \Delta y) + iV(x + \Delta x, y + \Delta y)] - [U(x, y) + iV(x, y)]$$



We have two paths to move on a plane starting from x-axis and y-axis as you choose.

In eqn (1): first we choose path-I where $\Delta y \rightarrow 0$ and $\Delta x \rightarrow 0$. After Δy is zero.

$\Delta z = \Delta x$ the eqn (1) becomes

$$f'(z) = \lim_{\Delta x \rightarrow 0} \frac{U(x + \Delta x, y) - U(x, y)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{V(x + \Delta x, y) - V(x, y)}{\Delta x}$$

$$= U_x + iV_x \quad \text{--- (ii)}$$

Exercise-2.3 Q. 5-9

We choose path-II. Where $\Delta x \rightarrow 0$ in eqⁿ

(i). After Δx is zero.

$\Delta z = i\Delta y$ the eqⁿ (i) becomes.

$$f'(z) = \lim_{\Delta y \rightarrow 0} \frac{U(x, y+\Delta y) - U(x, y)}{i\Delta y} + \lim_{\Delta y \rightarrow 0} \frac{U(x, y+\Delta y) - U(x, y)}{i\Delta y}$$

$$= \frac{1}{i} U_y + \frac{U_y}{-i}$$

$$= -iU_y + U_y \rightarrow (iii)$$

From eqⁿ (ii) and (iii)

$$f'(z) = f'(z)$$

$$U_x + iU_y = -iU_y + U_y$$

Equating real and im. part;

$$U_x = U_y \quad \text{and} \quad -U_y = U_x$$

$$\therefore U_x = U_y \quad \text{and} \quad U_y = -U_x$$

✓ Hence proved.

Ex 4

$$i) \int 2x(3x-2)dx + \{x^2 - (3x-2)^2\} 3dx$$

$$\int (6x^2 - 4x)dx + \int x^2 - (9x^2 - 12x + 4)3dx$$

$$\int (6x^2 - 4x)dx + \int \{3x^2 - 27x^2 + 36x - 12\}dx$$

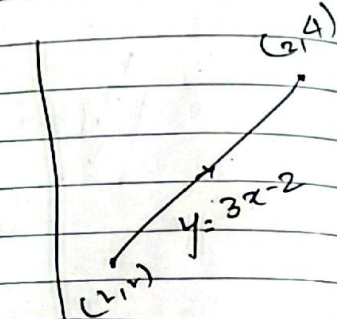
$$\int \{6x^2 - 4x + 3x^2 - 27x^2 + 36x - 12\}dx$$
$$= \int (-18x^2 + 32x - 12)dx$$

Complex Integration [8-marks]

Line integral in complex form.

The st. line joining the points $1+i$ and $2(1+2i)$ for $\int z^2 dz$.

$\Rightarrow$ here, $y = 3x - 2$
 $dy = 3dx$



$$\int_C z^2 dz$$

$$= \int (x+iy)^2 (dx+idy)$$

$$= \int (x^2 + 2xyi - y^2) (dx+idy)$$

$$= \int [x^2 + 2x(3x-2)i - (3x-2)^2] dx - 2x(3x-2)3dx$$

$$= \int [x^2 + 2xyi - y^2] dx + \int [x^2 + 2xyi - y^2] idy$$

$$= \int [x^2 + 2x(3x-2)i - (3x-2)^2] dx + \int [x^2 + 2xy(3x-2)i - (3x-2)^2 \cdot i] 3dx$$

$$= \int [x^2 + 6x^2i - 4xi - (9x^2 - 12x + 4)] dx$$

$$+ \int [x^2 + 6x^2i - 4xi - (9x^2 - 12x + 4)] i 3dx$$

$$= \int [-8x^2 + 12x - 4 + 6x^2i - 4xi] dx$$

$$+ \int [3x^2i - 18x^2 + 12x - (27x^2i - 36xi + 12i)] dx$$

$$= \int [-8x^2 + 12x - 4 - 18x^2 + 12x] dx$$

$$+ i \int [6x^2 - 4x + 3x^2 - 27x^2 + 36x - 12] dx$$

$$= \int_1^2 [-26x^2 + 24x - 4] dx + i \int_1^2 [-8x^2 + 32x - 12] dx$$

$$= -\frac{86}{3} + i(-6)$$

$$= -\left(\frac{86}{3} + i6\right)$$

Q. # Show that $\oint_C \frac{dz}{z} = 2\pi i$, Where C is a unit circle.

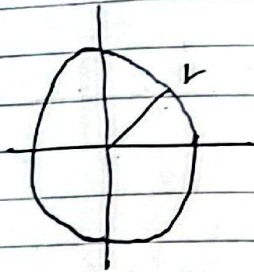
$\Rightarrow$ Given C is a unit circle so, we can put $z = e^{i\theta}$.

$$dz = i e^{i\theta} d\theta$$

Now,

$$\begin{aligned} \oint_C \frac{dz}{z} &= \int_0^{2\pi} \frac{i e^{i\theta} d\theta}{e^{i\theta}} \\ &= i \int_0^{2\pi} d\theta \\ &= i [0]_0^{2\pi} \\ &= 2\pi i \end{aligned}$$

proved



Complex Integration :

Q. Find $\oint_C \frac{dz}{z-a}$, Where C is the circle $|z-a|=r$.

$\Rightarrow$ Let $I = \oint_C \frac{dz}{z-a}$ Where circle is $|z-a|=r$.

for the circle we have to put $z = a + r e^{i\theta}$

$$dz = r i e^{i\theta} d\theta$$

then,

$$\begin{aligned} I &= \int \frac{r i e^{i\theta} d\theta}{r e^{i\theta}} \\ &= i \int d\theta \end{aligned}$$

the limit varies in circle $0 \leq \theta \leq 2\pi$

$$\begin{aligned} &= i \int_0^{2\pi} d\theta \\ &= 2\pi i \end{aligned}$$

#

Ex. 4.1.1

Q. $\int \sin^2 z \, dz$ $C: -\pi i$ along $|z| = \pi$ to πi in the half plane.

$$\Rightarrow \int_{-\pi i}^{\pi i} \sin^2 z \, dz$$

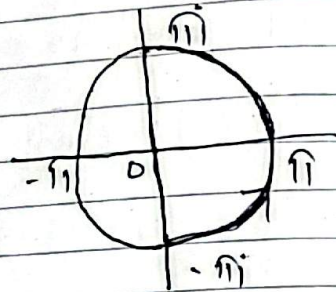
$$= \int_{-\pi i}^{\pi i} \frac{1 - \cos 2z}{2} \, dz$$

$$= \frac{1}{2} \left[\int_{-\pi i}^{\pi i} 1 \, dz - \int_{-\pi i}^{\pi i} \cos 2z \, dz \right]$$

$$= \frac{1}{2} \left[2\pi i - \left[\frac{\sin 2z}{2} \right]_{-\pi i}^{\pi i} \right]$$

$$= \pi i - \frac{1}{4} [\sin 2 \cdot \pi i + \sin 2 \cdot \pi i]$$

$$= \pi i$$



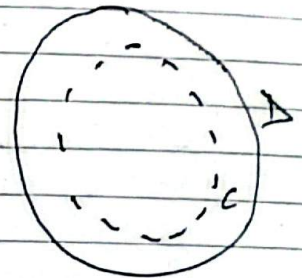
Cauchy's Integral theorem:

Statement: If $f(z)$ is analytic in a simple connected domain D , for every simple closed path C in D then,

$$\oint_C f(z) \, dz = 0$$

or,

If $f(z)$ is analytic and $f(z)$ is continuous at every point inside and on a simple closed curve C , then, $\oint_C f(z) \, dz = 0$.



Q. Evaluate the integral $\oint_C \frac{1}{z^2-3} \, dz$ if C is $|z| = 1$.

$\Rightarrow$ Now, we have by Cauchy's integral formula,

testing is $f(z)$ analytic or not analytic at that point. $z^2-3 = 0$

$$\Rightarrow z = \frac{3}{2} = 1.5$$

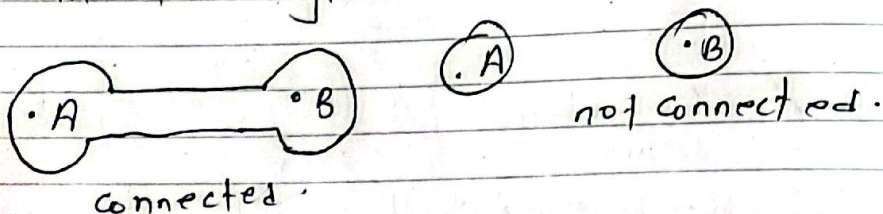
Which does not lie on the given domain $|z| = 1$. i.e. Circle with centre $(0,0)$ and radius 1.

But $z = 1.5$ do not contain the circle. So,

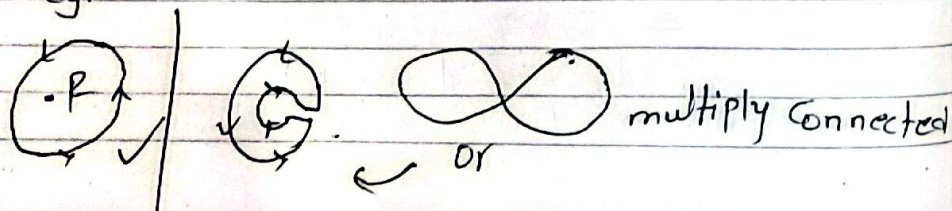
$$\oint \frac{dz}{z^2 - 3} = 0 \quad \#$$

Q. Define Connected region.

$\Rightarrow$ A Connected region is one in which any two points in it can be connected by a curve which lies entirely within the region.



2) Simply Connected region: A curve which does not cross itself is called Simply Connected region or curve.



#

Cauchy's Integral formula:

If $f(z)$ is analytic inside and on a curve C of simply connected region R and if 'a' is any point in C , then,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \quad ; \text{ integration orientating in counter clockwise.}$$

$$\text{i.e.} \quad \oint \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

Q. Evaluate the integral $\oint_C \frac{1}{z^2 + 3} dz$ where $C: |z| = 2$

$$\Rightarrow \text{Given } \oint_C \frac{1}{z^2 + 3} dz = \frac{1}{2(z + \frac{3}{2})}$$

for testing analyticity,

$$2(z + \frac{3}{2}) = 0$$

$$z = -\frac{3}{2} = -1.5 \text{ lies inside the circle.}$$

for this by C.I. f.

$$\oint_C \frac{1}{z(z+\frac{3}{2})} dz = \frac{1}{2} \int \frac{1}{(z+\frac{3}{2})} dz = \frac{1}{2} \int \frac{1}{(z-(-\frac{3}{2}))} dz$$

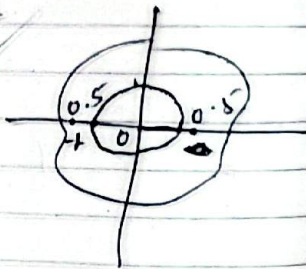
Here $f(z) = 1$, $a = -\frac{3}{2}$

Hence,

$$\frac{1}{2} \int \frac{1}{z-(-\frac{3}{2})} dz = \frac{1}{2} \times 2\pi i f(a)$$

$$= \pi i \times 1.$$

$$= \pi i$$



Q. Evaluate the integral $\oint_C \frac{3z^2+7z+1}{z+1} dz$ of

$C: |z| = \frac{1}{2}$ and $|z| = 2$.

$\Rightarrow$ Here, for analyticity test,

$$z+1=0$$

$z = -1$ does not lie on the circle.

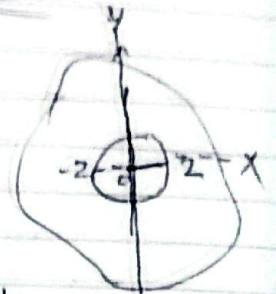
the.

$$\oint \frac{3z^2+7z+1}{z+1} dz = 0$$

again, for analyticity test,

$$z+1=0$$

$z = -1$ lies inside the circle



for by C.I. f.

$$\oint_C \frac{3z^2+7z+1}{z-(-1)} dz$$

Here $f(z) = 3z^2+7z+1$, $a = -1$.

Hence,

$$\int \frac{3z^2+7z+1}{z-(-1)} dz = 2\pi i f(a)$$

$$= 2\pi i [3(-1)^2 + (7 \times (-1)) + 1]$$

$$= 2\pi i (-3)$$

$$= -6\pi i$$

Integral: $\oint_C \log(1-z) dz$ where C : boundary of parallelogram. with vertices $\pm i, \pm(1+i)$

$\Rightarrow$ Here.

Given vertices,

$$z = \pm i \Rightarrow (0, 1), (0, -1)$$

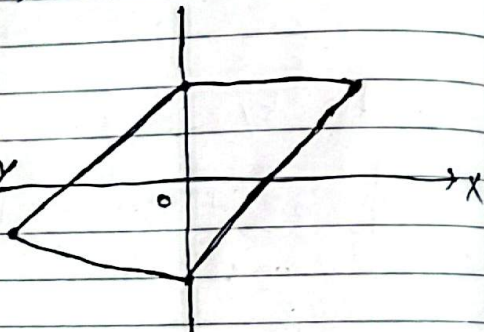
$$z = \pm(1+i) \Rightarrow (1, 1), (1, -1)$$

Now,

$$1-z=0 \Rightarrow z=1$$

which lies the boundary of the curve so,

$$\oint_C \log(1-z) dz = 0.$$



8-marks

Cauchy formula for Derivatives of an analytic function:

Statement: Let $f(z)$ be analytic funcⁿ inside and on a simple closed curve C enclosing a simply connected region R and if z_0 is any point on the domain of C then,

$$f'(z_0) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

or,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n \text{ is a positive integer}$$

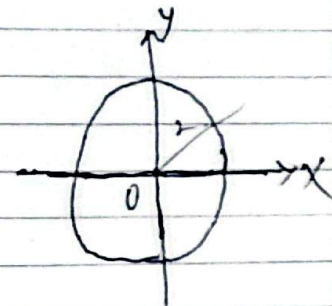
For example Integrate: $\oint_C \frac{\sin z}{z^4} dz$, where C is

Unit circle around the count clockwise direction.

$\Rightarrow$ The Given funcⁿ is not analytic at $z^4=0$

$$\Rightarrow z=0$$

$$\text{so, } z_0 = 0$$



$$y = \frac{1}{2} x^2$$

By Cauchy integral formula.

$$\oint_C \frac{\sin z}{z^4} dz = \frac{2\pi i}{3!} f'''(0) \quad \text{--- (1)}$$

$$\text{Now, } f(z) = \sin z$$

$$f'(z) = \cos z$$

$$f''(z) = -\sin z$$

$$f'''(z) = -\cos z$$

$$\text{at } f'''(0) = -\cos 0 = -1.$$

Hence eqn (1) becomes.

$$\begin{aligned} \oint_C \frac{\sin z}{z^4} dz &= \frac{2\pi i}{6} (-1) \\ &= -\frac{\pi i}{3} \end{aligned}$$

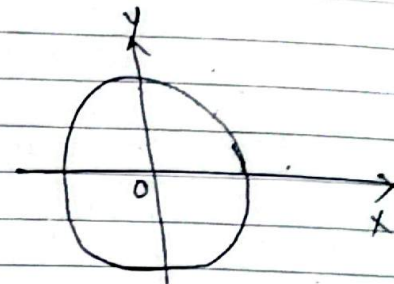
P.V

$$Q. \oint_C \frac{z^6}{(z^2-1)^6} dz \quad C: |z|=1$$

$\Rightarrow$ Given,

$$\oint_C \frac{z^6}{(z^2-1)^6} dz$$

$$\oint_C \frac{1}{2^6} \frac{z^6}{(z-\frac{1}{2})^6} dz$$



Now, the given funcⁿ is not analytic at

$$(z - \frac{1}{2})^6 = 0$$

$$z = \frac{1}{2}$$

So,

$$z_0 = \frac{1}{2}$$

By Cauchy Integral formula.

$$\oint_C \frac{1}{2^6} \frac{z^6}{(z-\frac{1}{2})^6} dz = \frac{2\pi i}{5!} f^{(5)}\left(\frac{1}{2}\right) \quad \text{--- (1)}$$

Now,

$$f(z) = z^6$$

$$f'(z) = 6z^5$$

$$f''(z) = 30z^4$$

$$f'''(z) = 120z^3, \quad f^{(4)}(z) = 360z^2$$

$$f^{(5)}(z) = 720z$$

at $z_0 = \frac{1}{2}$, the $f\left(\frac{1}{2}\right) = 720 \times \frac{1}{2}$
 $= 360$

Now, eqn (i) becomes,

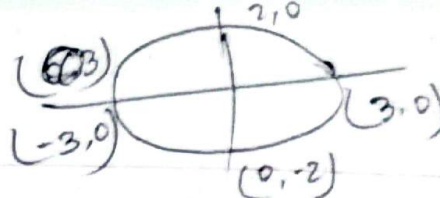
$$= \frac{1}{26} \frac{2\pi i}{5!} \times 360$$

$$= \frac{3\pi i}{32}$$

Exam.

Evaluate:

$$\oint_C \frac{6+z}{\left(z - \frac{\pi}{2}\right)^2} dz \quad \text{Where } C: 4x^2 + 9y^2 = 36$$



$\Rightarrow$ Given.

funcⁿ is not analytic at $\left(z - \frac{\pi}{2}\right)^2 = 0$

$$\Rightarrow z = \frac{\pi}{2} = 1.57$$

then,

$$z = z_0 = \frac{\pi}{2}$$

Now, $\oint_C \frac{6+z}{\left(z - \frac{\pi}{2}\right)^2} dz = \frac{2\pi i}{1!} f'\left(\frac{\pi}{2}\right)$, Where (i) $f(z) = 6+z$

$$f'(z) = -\operatorname{cosec}^2 z$$

$$\Rightarrow f'\left(\frac{\pi}{2}\right) = -\operatorname{cosec}^2 \frac{\pi}{2} = -1$$

From (i), we get,

$$\oint_C \frac{6+z}{\left(z - \frac{\pi}{2}\right)^2} dz = 2\pi i (-1) = -2\pi i$$

Q. Evaluate the integral: $\oint_C \frac{z+1}{z^3-2z^2} dz$, $C: |z|=1$.

$\Rightarrow$ Given,
funcⁿ is not analytic at

$$z^3 - 2z^2 = 0$$

$$z^2(z-2) = 0$$

$$z=0, z=2$$

then,

$$z=z_0=0 \text{ then, } \oint_C \frac{z+1}{(z-2)z^2} dz.$$

Now,

$$\oint \frac{z+1}{(z-2)z^2} dz = \frac{2\pi i}{1!} f'(0) \quad \text{--- (i)}$$

$$\text{Where, } f(z) = \frac{z+1}{z-2}$$

$$f'(z) = \frac{(z-2) \cdot 1 - (z+1)(1)}{(z-2)^2}$$

$$= \frac{(z-2) - (z+1)}{(z-2)^2} = \frac{z-2-z-1}{(z-2)^2} = \frac{-3}{(z-2)^2}$$

$$f'(0) = \frac{-3}{(0-2)^2} = \frac{-3}{4}$$

eqⁿ becomes,

$$\oint = 2\pi i \cdot \frac{-3}{4} = -\frac{3\pi i}{2}$$

(1.10) integrate

(a) $\oint_C \frac{\sinh z}{z^4} dz$ for $|z|=1$.

(b) $\oint_C \frac{z^3 e^z}{(z-\frac{1}{2})^3} dz$ for $|z|=1$

(c) $f(z) = \frac{\log z}{(z-2)^2}$ $C: |z-3|=2$

(d) $f(z) = \frac{2z^3-3}{2(z-1-i)^2}$ $C: |z|=2$ Counter clockwise and $|z|=1$ clockwise.

a) Given, the funcⁿ is not analytic at.

$$z^4 = 0$$

$$z = 0$$

then

$$z = z_0 = 0$$

Now,

$$\oint \frac{\sinh z}{z^4} dz = \frac{2\pi i}{3!} f'''(0) \quad \text{--- (i)}$$

$$\text{Where, } f(z) = \sinh z$$

$$f'(z) = \cosh z$$

$$f''(z) = \sinh z$$

$$f'''(z) = \cosh z$$

at

$$f'''(0) = \cosh 0 = 1.$$

eqⁿ (i) becomes,

$$\oint \frac{\sinh z}{z^4} dz = \frac{2\pi i}{3!} \cdot 1 = \frac{2\pi i}{6} = \frac{\pi i}{3} \neq$$

b) Given, the funcⁿ is not analytic at.

$$\left(z - \frac{1}{2}\right)^3 = 0$$

$$z = \frac{1}{2}$$

then,

$$z = z_0 = \frac{1}{2} = 0.5$$

$$\text{Now, } \oint \frac{z^3 e^z}{\left(z - \frac{1}{2}\right)^3} dz = \frac{2\pi i}{2!} f''(0.5) \quad \text{--- (i)}$$

$$\text{Where, } f(z) = z^3 e^z$$

$$f'(z) = z^3 \cdot e^z + 3e^z z^2$$

$$\begin{aligned} f''(z) &= z^3 \cdot e^z + 3z^2 e^z + 3[e^z \cdot 2z + z^2 e^z] \\ &= z^3 e^z + 3z^2 e^z + 6e^z z + 3z^2 e^z \end{aligned}$$

$$f''(0.5) = 7.625$$

eqⁿ (i) becomes,

$$\oint \frac{z^3 e^z}{\left(z - \frac{1}{2}\right)^3} dz = \frac{2\pi i}{2} \cdot 7.625 = 7.625\pi i \neq$$

c) The funcⁿ is not analytic at $|z-3|=2$.

$$(z-2)^2 = 0$$

$$z = 2$$

then, $z = z_0 = 2$ which lies ~~inside~~ the given domain.

Now,

$$\oint_C \frac{\log(z)}{(z-2)^2} dz = \frac{2\pi i}{1!} f'(2) \quad \text{--- (i)}$$

Where, $f(z) = \log z$

$$f'(z) = \frac{1}{z}$$

$$f'(2) = \frac{1}{2} = 0.5$$

Now, eqⁿ (i) becomes,

$$\oint_C \frac{\log(z)}{(z-2)^2} dz = 2\pi i \times 0.5 = \pi i$$

d) Given,

funcⁿ is not analytic at.

$$2(z-1-i)^2 = 0$$

$$\Rightarrow z = 0, z = 1+i = (1, 1)$$

Since,

$z = 0$ does not lie on the required domain but $z = 1+i$ lies on it.

So,

$$f(z) = \frac{2z^3 - 3}{z}, \quad z_0 = (1+i)$$

Now,

$$\oint_C \frac{2z^3 - 3}{(z-1-i)^2} dz = \frac{2\pi i}{1!} f'(1+i) \quad \text{--- (i)}$$

When,

$$f(z) = \frac{2z^3 - 3}{z} = \frac{2(6z^2) - (2z^3 - 3) \cdot 1}{z^2} = \frac{6z^3 - 2z^3 + 3}{z^2} = \frac{4z^3 + 3}{z^2}$$

$$f'(1+i) = \frac{4(1+i)^3 + 3}{(1+i)^2}$$

$$\text{Now } \oint_C \frac{2z^3 - 3}{(z-1-i)^2} dz = \frac{2\pi i (4(1+i)^3 + 3)}{(1+i)^2} \quad \text{--- (ii)}$$

Now. eqn (ii) solve.

Formula,

$$(i) \frac{1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

$$(ii) \frac{1}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$$

$$(iii) \frac{1}{(x-a)(x^2+b)} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+b)}$$

Sequence and series (7/8 marks)

$$f(z) = \sum_{i=1}^{\infty} z_i \rightarrow \text{series.}$$

Some Important formula for sequence and series.

Taylor series:

statement: If $f(z)$ be analytic in C with centre at 'a' and radius r_0 then each point z inside C , the series,

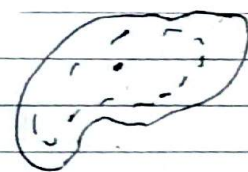
$$f(a) + f'(a)(z-a) + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)(z-a)^3}{3!} + \dots + \frac{f^{(n)}(a)(z-a)^n}{n!}$$

so,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n;$$

where, $a_n = \frac{f^{(n)}(a)}{n!}$

converse to $f(z)$



Note: Maclaurin series $f(0)$, $a=0$

If $a=0$ this gives Maclaurin expansion,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{f^{(n)}(0)}{n!}$$

Q.

Find the Taylor expansion of $\log(1+z)$ at $z=1$

$$\Rightarrow f(z) = \log(1+z) \Rightarrow f(1) = \ln 2.$$

$$f'(z) = \frac{1}{1+z} \Rightarrow f'(1) = \frac{1}{2}$$

$$f''(z) = \frac{-1}{(1+z)^2} \Rightarrow f''(1) = -\frac{1}{4}$$

$$f'''(z) = \frac{2}{(1+z)^3} \Rightarrow f'''(1) = \frac{2}{8} = \frac{1}{4}$$

We know that,

$$f(z) = f(a) + f'(a) \frac{(z-a)}{1!} + f''(a) \frac{(z-a)^2}{2!} + \dots$$

$$\Rightarrow \log(1+z) = \ln 2 + \frac{1}{2}(z-1) + \frac{(-1)}{4} \frac{(z-1)^2}{2!} + \dots$$

Some Imp formula for Maclaurin's expansion.

$$① e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!} + \dots + |z| < \infty \quad (\text{finite})$$

$$2) \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + \frac{(-1)^{n+1} z^{2n+1}}{(2n+1)!} + \dots$$

$$3) \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + \frac{(-1)^n z^{2n}}{(2n)!} + \dots$$

$$4) \frac{1}{1-z} = 1 + z + z^2 + \dots + z^n + \dots \quad |z| < 1$$

$$5) \frac{1}{1+z} = (z+1)^{-1} = 1 - z + z^2 - z^3 + \dots + (-1)^n z^n + \dots \quad |z| < 1$$

Q. Find the Taylor expansion about $z=0$ of
 $f(z) = \frac{z}{(z+1)(z-3)}$

$\Rightarrow$ The given funcⁿ is not analytic at
 $(z+1)(z-3) = 0$
 $\therefore z = -1, 3$

By using partial fraction,

$$\frac{z}{(z+1)(z-3)} = \frac{A}{(z+1)} + \frac{B}{(z-3)}$$

$$\Rightarrow z = Az - 3A + Bz + B$$

$$z = (A+B)z + (-3A+B)$$

Equating,

$$A+B=1$$

$$-3A+B=0$$

$$A=1-B \quad \text{---} \textcircled{*} \quad -3(1-B)+B=0$$

$$A = 1 - \frac{3}{4}$$

$$-3 + 3B + B = 0$$

$$4B = 3$$

$$\therefore A = \frac{1}{4}$$

$$\therefore B = \frac{3}{4}$$

$$\Rightarrow f(z) = \frac{1}{4} \frac{1}{(z+1)} + \frac{3}{4} \frac{1}{(z-3)}$$

$$= \frac{1}{4} \frac{1}{(z+1)} + \frac{3}{4} \frac{1}{(-3)(1-\frac{z}{3})}$$

$$= \frac{1}{4} \frac{1}{(1+z)} - \frac{1}{4} \frac{1}{(1-\frac{z}{3})}$$

$$= \frac{1}{4} [1 - z + z^2 - z^3 + \dots] - \frac{1}{4} [1 + (\frac{z}{3}) + (\frac{z}{3})^2 + \dots]$$

Q. Find the Taylor series expansion of a funcⁿ
 $\frac{z^2-1}{(z+2)(z+3)}$ in the region $|z| < 2$

$\Rightarrow$ The given funcⁿ, $\frac{z^2-1}{(z+2)(z+3)} = f(z)$

$$\Rightarrow f(z) = 1 + \frac{(-5z+7)}{(z+2)(z+3)}$$

for, $\frac{-5z+7}{(z+2)(z+3)} = \frac{A}{(z+2)} + \frac{B}{(z+3)}$

$$-5z - 7 = A(z+3) + B(z+2)$$

$$\therefore A = \underline{3} = 3$$

$$\therefore B = \frac{8}{-1} = -8$$

ନେଉ,

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)}$$

$$\Rightarrow \frac{2}{2} \mid \leq h. \quad \text{for } |z| < 2$$

$$= 1 + \frac{3}{2} \cdot \frac{1}{\left(1 + \frac{2}{2}\right)} - \frac{8}{3} \cdot \frac{1}{\left(1 + \frac{2}{3}\right)}$$

$$= 1 + \frac{3}{2} \left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \dots \right] - \frac{8}{3} \left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \dots \right]$$

#

~~§~~. LAURENT'S SERIES.

Let C_1, C_2 be two concentric circles $|z-a| < r_1$ and $|z-a| = r_2$ where $r_1 < r_2$. Let $f(z)$ be analytic on C_1 and C_2 and in the annular region R between them. Then for any point z in R .

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

Where,

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{1-n}} dz$$

$2 = 0, (-1, 1)$

V.V.I.M.I

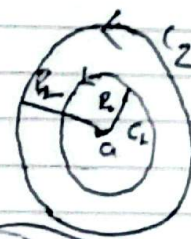
$$I = A(7-2)(7+1) + B(2)(7+1) + C(2)(7-2)$$

$$P = V$$

$$z C = -4e \quad B = 1/2$$

$$(z-2)(z-1)$$

$$2^2 - 2 - 2 \cdot 2 + 2$$



Q. Find the Laurent's series expansion of $\frac{1}{z^2+3z+2}$ in the region $1 < |z| < 2$.

$\Rightarrow$ Given, $f(z) = \frac{1}{z^2+3z+2} = \frac{1}{(z+1)(z+2)}$

By use partial fraction,

$$\frac{1}{z^2+3z+2} = \frac{A}{(z+1)} + \frac{B}{(z+2)}$$

$$1 = A(z+2) + B(z+1)$$

$$\therefore A = 1$$

$$\therefore B = -1$$

Now, $f(z) = \frac{1}{(z+1)} - \frac{1}{(z+2)} ; 1 < |z|, |z| < 2$

$$\Rightarrow \frac{1}{|z|} < 1, \frac{|z|}{2} < 1$$

$$= \frac{1}{z(1+\frac{1}{z})} - \frac{1}{2(1+\frac{z}{2})}$$

$$= \frac{1}{z} \left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots \right] - \frac{1}{2} \left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \dots \right]$$

$$= \frac{1}{z} \left[\left(1 + \frac{1}{z}\right)^{-1} \right] - \frac{1}{2} \left[\left(1 + \frac{z}{2}\right)^{-1} \right]$$

$$= \frac{1}{z} \left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots \right] - \frac{1}{2}$$

$$\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \dots \right]$$

#

Expand the Laurent series expansion
 $f(z) = \frac{z}{(z-1)(z-3)}$, valid for (a) $1 < |z| < 3$
 (b) $0 < |z-1| < 2$

$\Rightarrow$ Given.
 funcⁿ is not analytic at $(z-1)(z-3) = 0$
 $\Rightarrow z = 1, 3$
 By partial fraction.

$$\frac{z}{(z-1)(z-3)} = \frac{A}{(z-1)} + \frac{B}{(z-3)}$$

$$z = A(z-3) + B(z-1)$$

$$\therefore A = -1/2$$

$$B = 3/2$$

$$\Rightarrow \frac{z}{(z-1)(z-3)} = \frac{-1/2}{(z-1)} + \frac{3/2}{(z-3)}$$

$$= -\frac{1}{2} \frac{1}{(z-1)} + \frac{3}{2} \frac{1}{(z-3)} \quad \text{--- (i) (b)}$$

By condition,

$$\text{(a) } 1 < |z| < 3 \Rightarrow 1 < |z| \text{ and } |z| < 3$$

$$\Rightarrow \frac{1}{|z|} < 1 \text{ and } \left| \frac{z}{3} \right| < 1$$

Then eqⁿ (i), we get.

$$= -\frac{1}{2} \frac{1}{z(1-\frac{1}{z})} + \frac{3}{2} \frac{1}{3(\frac{z}{3}-1)}$$

$$= -\frac{1}{2z} \frac{1}{(1-\frac{1}{z})} - \frac{1}{2} \frac{1}{(1-\frac{z}{3})}$$

$$= -\frac{1}{2z} \left[1 - \frac{1}{z} \right]^{-1} - \frac{1}{2} \left[1 - \frac{z}{3} \right]^{-1}$$

$$= -\frac{1}{2z} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right] - \frac{1}{2}$$

$$\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \left(\frac{z}{3}\right)^3 + \dots \right]$$

By condition,

$$0 < |z-1| < 2 \Rightarrow \text{--- (b)}$$

$$0 < |z-1| \text{ and } |z-1| < 2$$

PV

Exam.

Put $z-1 = u \Rightarrow z = u+1$

then $0 < |u| < 2 \Rightarrow \left| \frac{u}{2} \right| < 1$

then.

$$f(z) = -\frac{1}{2} \left(\frac{1}{z-1} \right) + \frac{3}{2} \left[\frac{1}{z-3} \right]$$

$$= -\frac{1}{2} \left[\frac{1}{u} \right] + \frac{3}{2} \left[\frac{1}{u-2} \right]$$

$$= -\frac{1}{2} \left[\frac{1}{u} \right] + \frac{3}{2} \left[\frac{1}{u-2} \right]$$

$$= -\frac{1}{2} \frac{1}{u} + \frac{3}{2} (-2) \left[\frac{1}{1 - \frac{u}{2}} \right]$$

$$= -\frac{1}{2u} - \frac{3}{4} \left[1 - \frac{u}{2} \right]^{-1}$$

$$= -\frac{1}{2u} - \frac{3}{4} \left[1 + \frac{u}{2} + \left(\frac{u}{2} \right)^2 + \left(\frac{u}{2} \right)^3 + \dots \right]$$

$$= -\frac{1}{2(z-1)} - \frac{3}{4} \left[1 + \left(\frac{z-1}{2} \right) + \left(\frac{z-1}{2} \right)^2 + \left(\frac{z-1}{2} \right)^3 + \dots \right]$$

#

Q. Expand the series $f(z) = \frac{1}{z^2-3z+2}$ for the region $|z| < 1$, $1 < |z| < 2$, $|z| > 2$, $|z-1| < 2$.

$\Rightarrow$ Given

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-2)(z-1)}$$

By partial fraction,

$$\frac{1}{(z-2)(z-1)} = \frac{A}{(z-2)} + \frac{B}{(z-1)}$$

$$1 = A(z-1) + B(z-2)$$

$$\therefore A = 1$$

$$\therefore B = -1$$

Then, $\frac{1}{(z-2)(z-1)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$ 

By condition, $|z| < 1 \Rightarrow \left| \frac{z}{1} \right| < 1$

The eqⁿ (i), we get,

$$= \frac{1}{(z-2)(1-\frac{z}{2})} - \frac{1}{(z-1)(1-\frac{z}{2})}$$

$$= \frac{1}{(z-2)} \left(1-\frac{z}{2}\right)^{-1} + \frac{1}{(z-1)} \left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z-2} \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right] + \frac{1}{z-1} \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right]$$

Again, $1 < |z| < 2$

$$\Rightarrow \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1$$

the eqⁿ (i), we get,

$$= \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{2} \left(1-\frac{z}{2}\right)^{-1} - \left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{2} \left(1-\frac{z}{2}\right)^{-1} - \left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{2} \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right] - \left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\right]$$

Again, $|z| > 2 \Rightarrow \left|\frac{z}{2}\right| < 1$, Now eqⁿ (i).

$$\Rightarrow \frac{1}{(z-2)} - \frac{1}{(z-1)} = \frac{1}{z} \left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z} \left(1-\frac{1}{z}\right)^{-1}$$

$$\Rightarrow \frac{1}{z} \left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z} \left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} \left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \dots\right] - \frac{1}{z} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\right]$$

Again, put $u = z-1 \Rightarrow |u| < 2 \Rightarrow \left|\frac{u}{2}\right| < 1$

$$= \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{(z-1-1)} - \frac{1}{u} \Rightarrow \frac{1}{(u-1)} - \frac{1}{u}$$

$$= \frac{1}{(1-\frac{u}{2})} - \frac{1}{u} \Rightarrow \left(1-\frac{u}{2}\right)^{-1} - \frac{1}{u}$$

$$= \left[1 + \frac{u}{1} + \left(\frac{u}{1}\right)^2 + \frac{u^3}{1} + \dots \right] - \frac{1}{u}$$

$$= \left[1 + \frac{z-1}{1} + \left(\frac{z-1}{1}\right)^2 + \frac{(z-1)^3}{1} + \dots \right] - \frac{1}{z-1}$$

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Q. (4-11)

11.

Zeros and Singularities $\rightarrow$ [8-marks]

1. Zeros of an analytic funcⁿ:

If a funcⁿ $f(z)$ is analytic in a region R , is zero at a point $z = z_0$ in R , then z_0 is called zero of $f(z)$.

(a) Simple zero: If $f(z_0) = 0$ and $f'(z_0) \neq 0$ then $z = z_0$ is called zero of $f(z)$ in first order.

Example: $f(z) = \frac{z^2 + 1}{1 - z^2}$ for simple zero.

$$z^2 + 1 = 0$$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm i$$

$\Rightarrow z = -i, i$ are the simple zeros.

2. zeros of order n : If $f(z_0) = f'(z_0) = \dots = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$ then z_0 is called zero of order n .

Example. find the zeros of the funcⁿ $f(z) = \sin\left(\frac{1}{2a}z\right)$.

$\Rightarrow$ This is sine function the general solution of sine function for $\sin \theta = 0$ is $\theta = n\pi$

$$\sin \theta = 0 \Rightarrow \theta = n\pi.$$

$$\Rightarrow \sin\left(\frac{1}{z-a}\right) = 0$$

$$\frac{1}{z-a} = n\pi \quad \left[\because n = \pm 1, \pm 2, \dots \right]$$

$$\text{or, } 1 = n\pi(z-a)$$

$$\therefore z = \left(a + \frac{1}{n\pi}\right) \neq$$

Q. Find the zeros of the funcⁿ $\frac{\sin z - z}{z^3}$.

$$\Rightarrow f(z) = \frac{\sin z - z}{z^3}$$

$$= \left[\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) - z \right] \frac{1}{z^3}$$

$$= -\frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$= -\frac{1}{3!} + \frac{z^2}{5!} - \dots$$

$\Rightarrow \frac{\sin z - z}{z^3} \neq 0$, so it does not have a zero.

Singularities points:

A point $z = z_0$ at which a funcⁿ $f(z)$ fails to be analytic is called the singular points a singularity of $f(z)$.

$$\text{Ex. } f(z) = \frac{z}{z-3}$$

Here, $z=3$ is a singularity of $f(z)$.

[$\therefore$ Note: Denominator = 0]

[Note: zeros check: Numerator = 0]

There are different types of singularities.

(a) isolated singularities: A point $z = z_0$ is said to be isolated singularities of $f(z)$ if $f(z)$ is not analytic at $z = z_0$ and there exist a neighbourhood of $z = z_0$ containing no other singularity.

Ex.

$$f(z) = \frac{1}{z}$$

⑥ Removable singularities: If the principal part of $f(z)$ contains no terms, i.e. $b_n = 0$ for all n then that $z = z_0$ is known as removable singularities of $f(z)$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

or,

A singular point $z = z_0$ is called a removable singularities of $f(z)$ if $\lim_{z \rightarrow z_0} f(z)$ exists finitely.

For Ex. Test the nature of singularity at $z = 0$.

$$f(z) = \frac{\sin z - z}{z^3}$$

By def?

$$\Rightarrow \lim_{z \rightarrow z_0} f(z) = \text{finite.}$$

$$\Rightarrow \lim_{z \rightarrow 0} \frac{\sin z - z}{z^3} \left[\frac{0}{0} \right]$$

$$\Rightarrow \lim_{z \rightarrow 0} \frac{\cos z - 1}{3z^2} \left[\frac{0}{0} \right]$$

$$\Rightarrow \lim_{z \rightarrow 0} \frac{-\sin z}{6z} \left[\frac{0}{0} \right]$$

$$\Rightarrow \lim_{z \rightarrow 0} \frac{-\cos z}{6} = -\frac{1}{6} \neq$$

This is removable singularities.

$$\text{Note: } f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

≠ Poles: If we can find a positive integer n such that $\lim_{z \rightarrow a} \frac{(z - a)^n}{f(z)} \neq 0$ then $z = a$

is called a pole of order n for $f(z)$.

or,

An analytic funcⁿ $f(z)$ with a singularity at $z = a$ if $\lim_{z \rightarrow a} f(z) = \infty$ then $z = a$

is a pole of $f(z)$. If pole of order one is called simple pole.

Ex.

$$\frac{1}{(z-4)^2 (z-3)^3 (z-1)}.$$

$\Rightarrow$ In this question, there are different poles.

Where.

$z=1$ is a simple pole
 $z=3$ is a pole of order 3 [power]
 $z=4$ is a pole of order 2. [power]

Essential Singularity:
If the principal part of $f(z)$ contains an infinite number of non zero terms then $z=z_0$ is known as essential singularities.

$$f(z) = e^{1/z} = 1 + \frac{(\frac{1}{z})}{1} + \frac{(\frac{1}{z})^2}{2!} + \dots$$

has $z=0$ as an essential singularity.
and

$f(z) = e^{\frac{1}{z-4}}$ has $z=4$ as essential singularity.

Note: The removable singularity and poles are isolated singularities. But the essential singularity is either isolated or non-isolated singularity.

C/W. Test the singularity of

$$f(z) = \frac{e^{1/z}}{(z-a)^2}$$

$$\text{Here, } (z-a)^2 = 0$$

$$z=a$$

Classify the singularity of $f(z) = \frac{z - \sin z}{z^3}$
for the singularity of $f(z)$.

$$z^3=0$$

The pole is $z=0$ of order 3.

How to find residues of funcⁿ $f(z)$?

→ Residue of a pole of order m :

If $z=z_0$ is a pole of order m , a simple formula to determine the residue is given by.

$$(i) \quad b_1 = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$$

(ii) Residue at a pole.

If $z=z_0$ is simple pole.

$$\text{i.e. } m=1 \text{ then, } b_1 = \lim_{z \rightarrow z_0} (z-z_0) f(z)$$

$$\text{i.e. } \text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} (z-z_0) f(z).$$

ψ

(iii) If $z=z_0$ is a simple pole of $f(z)$

and if $f(z) = \frac{\phi(z)}{\psi(z)}$ then

$$\text{Res}[f(z), z_0] = \frac{\phi(z_0)}{\psi'(z_0)} \quad \#$$

Residues of $f(z)$:

Example: find the value of $f(z) = \frac{1-e^{2z}}{z^3}$

⇒ Since $f(z) = \frac{1-e^{2z}}{z^3}$

here $f(z)$ is not analytic at $z=0$ is a pole of order 3

so, $m=3$.

then

$$\text{Res}[f(z), z_0] = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$$

$$\Rightarrow \text{Res}[f(z), 0] = \frac{1}{2!} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} [(z-0)^3 \frac{1-e^{2z}}{z^3}]$$

$$= \frac{1}{2!} \lim_{z \rightarrow 0} \left[\frac{d}{dz} (0 - 2e^{2z}) \right] \quad \checkmark$$

$$= \lim_{z \rightarrow 0} \frac{1}{2} [-4e^{2z}]$$

$$= -2e^{2 \times 0}$$

$$= -2 \quad \text{Ans} \quad \#$$

Q. find the residue of $\frac{1}{z^2 e^z}$

⇒ Given.

$$f(z) = \frac{1}{z^2 e^z} = \frac{1}{z^2} e^{-z}$$

$$= \frac{1}{z^2} \left[1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots \right]$$

$$= \frac{1}{z^2} - \frac{1}{z!} + \frac{1}{2!} - \frac{z}{3!} + \frac{z^2}{4!} - \dots$$

The residue of $f(z) = -1$ $\#$

[∵ residue is denominator में power 1 शेष
Coefficient is Answer.] Take above.

Example, $-\left(\frac{1}{z}\right) = -1$. [Pole नहीं है
मात्र formula use]

Q. Find the res. of $\frac{z^2}{z^2+a^2}$ at $z=ia$

$\Rightarrow$ Here, $f(z) = \frac{z^2}{z^2+a^2}$

The poles of this funcⁿ $z^2+a^2=0$
 $\Rightarrow z = \pm ia$
 $\Rightarrow z = +ia, -ia$

for the res. of $f(z)$ at $z=ia \Rightarrow \text{Res}[f(z), ia]$

Here, res. $f(z)$ at $z=ia$

$$\Rightarrow b_1 = \lim_{z \rightarrow ia} (z-ia)' f(z)$$

$$= \lim_{z \rightarrow ia} \left[(z-ia) \frac{z^2}{(z-ia)(z+ia)} \right]$$

$$= \lim_{z \rightarrow ia} \left[\frac{z^2}{(z+ia)} \right]$$

$$= \frac{(ia)^2}{2ia} = \frac{ia}{2}$$

$\frac{ia}{2}$

clw

find the residue of: (a) $\frac{\sin 2z}{z^6}$ (b) $\frac{1}{1-e^z}$

(a) $\frac{1}{(z^2+a^2)^2}$ at $z=ia$

$\Rightarrow$ Given,

$$f(z) = \frac{\sin 2z}{z^6}$$

Here, $f(z)$ is not analytic at $z=0$ is a pole of order 6

so, $m=6$
then,

$$\text{Res}[f(z), z_0] = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$$

$$\text{Res}[f(z), 0] = \frac{1}{5!} \lim_{z \rightarrow 0} \frac{d^5}{dz^5} \left[\frac{\sin 2z}{z^6} (z-0)^6 \right]$$

$$= \frac{1}{120} \lim_{z \rightarrow 0} \left[\frac{d^4}{dz^4} 2 \cos 2z \right]$$

$$= \frac{1}{120} \lim_{z \rightarrow 0} 2 \left[\frac{d^3}{dz^3} -2 \sin 2z \right]$$

$$= \frac{1}{20} \lim_{z \rightarrow 0} (-4) \left[\frac{d^2}{dz^2} 2 \cos 2z \right]$$

$$= \frac{1}{120} \lim_{z \rightarrow 0} (-8) \left[\frac{d}{dz} (z \sin 2z) \right]$$

$$= \frac{1}{120} \lim_{z \rightarrow 0} (16) \cdot 2 \cos 2z$$

$$= \frac{32}{120} \lim_{z \rightarrow 0} \cos 2z$$

$$= \frac{32}{120} \times 1$$

$$= \frac{4}{15}$$

(b) here $f(z) = \frac{1}{1-e^z}$

(c) here $f(z) = \frac{1}{(z^2+a^2)^2}$ at $z=ia$

The pole of this func $z^2+a^2=0$

$$\Rightarrow z = \pm ia$$

$$\Rightarrow z = +ia, -ia$$

for the res. of $f(z)$ at $z=ia$

$$m=2.$$

$$\Rightarrow b_1 = \frac{1}{(2-1)!}$$

$$\Rightarrow \text{Res. } [f(z), ia] \text{ at } z=ia$$

Drawing Compulsory

Cauchy Residue Theorem:

Statement: let $f(z)$ be analytic inside a simple closed path C and on C except for finitely many singular points $z_1, z_2, \dots, z_k$ in C . Then the integral of $f(z)$ taken counterclockwise around C and equals to $2\pi i$ times of the sum of residue of $f(z)$ at $z_1, z_2, \dots, z_k$. That is.

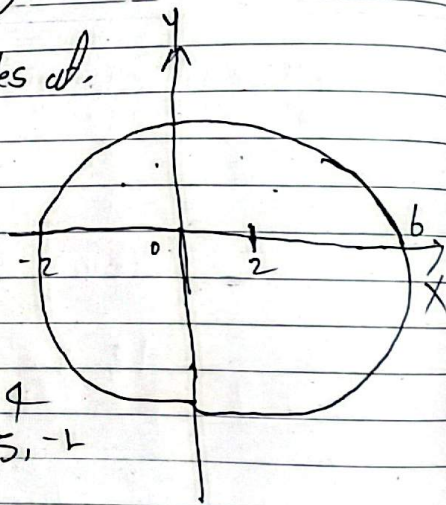
$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^k \text{res.}_{z=z_j} (f(z))$$

Q: Integrate $\oint_C \left(\frac{z-23}{z^2-4z-5} \right) dz$ $C: |z-2|=4$.

$\Rightarrow$ Given, funcⁿ has poles at $z^2-4z-5=0$

$$z = 5, -1$$

Since, the given circle $|z-2|=4$ with centre $(2,0)$ and radius 4 lies both points $z = 5, -1$ inside C .



So, there exists, residue over these points.

The residue at $z = -1$.

$$\begin{aligned} \text{res}[f(z), -1] &= \lim_{z \rightarrow -1} [(z+1) f(z)] \\ &= \lim_{z \rightarrow -1} \left[(z+1) \frac{z-23}{(z+1)(z-5)} \right] \\ &= \frac{-24}{-6} = 4 \end{aligned}$$

Again,

The residue at $z = 5$,

$$\begin{aligned} \text{res}[f(z), 5] &= \lim_{z \rightarrow 5} [(z-5) f(z)] \\ &= \lim_{z \rightarrow 5} \left[(z-5) \frac{z-23}{(z+1)(z-5)} \right] \\ &= \frac{-18}{6} = -3 \end{aligned}$$

$$\text{Now, } \oint_C \left(\frac{z-23}{z^2-4z-5} \right) dz = 2\pi i [4 + (-3)] = 2\pi i$$

↓
sum of residue: #

C/W

Q. find the residue of the following Counterclockwise
 1) Evaluate integral.

$$\oint_C \frac{dz}{z^8(z+4)} \quad C: \textcircled{a} |z|=2 \quad \textcircled{b} |z+2|=3$$

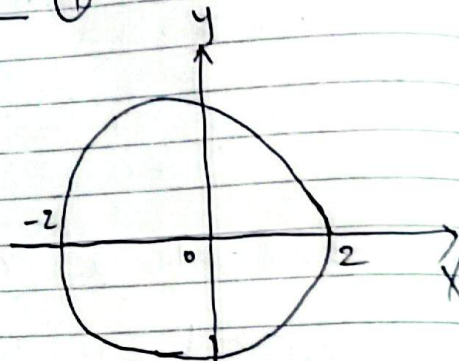
⇒ Here,

$$\oint_C \frac{dz}{z^8(z+4)}$$

① $C: |z|=2$

funcⁿ has poles at
 $z^8(z+4)=0$

$z=0, -4$ Since, the given circle $|z|=2$ with centre $(0,0)$ and radius 2 lies only points 0 inside C.



$$\begin{aligned} \text{Res}[f(z), 0] &= \lim_{z \rightarrow 0} \left[(z-0)^8 f(z) \right] \\ &= \lim_{z \rightarrow 0} \left[\frac{1}{z^8(z+4)} \right] = \end{aligned}$$

$$\text{Res}[f(z), 0] = \lim_{z \rightarrow 0} \left(\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-0)^m \frac{1}{z^8(z+4)} \right] \right)$$

$$= \lim_{z \rightarrow 0} \frac{1}{(8-1)!} \frac{d^7}{dz^7} \left[\frac{1}{z+4} \right]$$

$$= \frac{1}{7!} \lim_{z \rightarrow 0} \frac{d^7}{dz^7} \left[\frac{1}{z+4} \right] \quad \text{--- } \textcircled{X}$$

Now, $\frac{d}{dz} (z+4)^{-1} = -(z+4)^{-2}$ ~~and $z=0, \frac{d}{dz} \left[\frac{1}{z+4} \right] = -\frac{1}{(z+4)^2}$~~

$$\frac{d^2}{dz^2} (z+4)^{-2} = 2(z+4)^{-3} \quad \text{--- } L$$

$$\frac{d^3}{dz^3} 2(z+4)^{-3} = -6(z+4)^{-4} \quad \text{--- } L$$

$$\frac{d^4}{dz^4} -6(z+4)^{-4} = 24(z+4)^{-5} \quad \text{--- } L$$

$$\frac{d^5}{dz^5} 24(z+4)^{-5} = -120(z+4)^{-6} \quad \text{--- } L$$

$$\frac{d^6}{dz^6} [-120(z+4)^{-6}] = 720(z+4)^{-7} \quad \text{--- } L$$

$$\frac{d^7}{dz^7} 720(z+4)^{-7} = -5040(z+4)^{-8} \quad \text{--- } L$$

0.0

Now, at $z=0$,

$$\frac{d^7}{dz^7} [z^8(z+4)^{-8}] = \frac{-5040}{(z+4)^8} = \frac{-5040}{4^8} (4)^{-8}$$

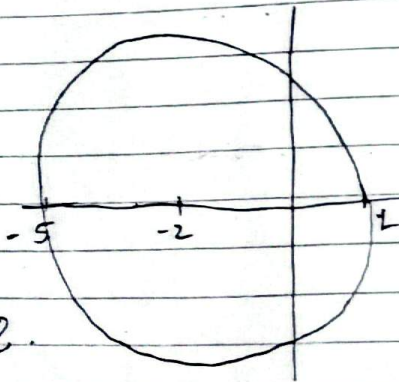
$$\text{Res}(f(z), 0) = \lim_{z \rightarrow 0} \frac{1}{7!} \left[\frac{-5040}{4^8} \right] \cdot \frac{1}{z^8(z+4)} = \frac{-5040}{4^8 \cdot 7!}$$

$$\oint_C \frac{dz}{z^8(z+4)} = 2\pi i \times (-4^{-8}) = \frac{-2\pi i}{2^{16}}$$

$$\frac{-2\pi i}{2^{16}} = \frac{(-\pi i)}{2^{15}}$$

$$(6) |z+2| = 3.$$

Since, the given
Circle $|z+2| = 3$
with centre $(-2, 0)$
lies both points
 $0, -4$ inside C .

Now, residue at $z=0$.

$$\text{Res}(f(z), 0) = -(4^{-8}).$$

1. D.

Again,

$$\text{Res}[f(z), -4] = \frac{1}{(7-1)!} \lim_{z \rightarrow -4} \frac{d^6}{dz^6} \left[(z+4)^{-8} \frac{1}{z^8(z+4)} \right]$$

$$= \frac{1}{7!} \lim_{z \rightarrow -4} \frac{d^7}{dz^7} \left[(z+4)^{-8} \frac{1}{z^8(z+4)} \right]$$

$$= \frac{1}{7!} \lim_{z \rightarrow -4} \frac{d^7}{dz^7} \left[\frac{(z+4)^{-7}}{z^8} \right]$$

$$\text{Now, } \frac{d}{dz} \left[\frac{(z+4)^{-7}}{z^8} \right] = \frac{-7(z+4)^{-8}}{z^8} - \frac{(z+4)^{-7} \cdot 8z^7}{z^{16}}$$

$$\text{Now, } \frac{d}{dz} \left[\frac{(z+4)^{-7}}{z^8} \right] = \frac{-7(z+4)^{-8}}{z^8} - \frac{(z+4)^{-7} \cdot 8z^7}{z^{16}}$$

$$= \frac{-7(z+4)^{-8}}{z^8} - \frac{8z^7(z+4)^{-7}}{z^{16}}$$

Unit IV The Fourier Integral [8+7=15 marks]

Fourier series: $-\infty$
 $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (-\pi \text{ to } \pi)$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Odd/even/neither odd nor even funcⁿ

Sine funcⁿ:-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (i)}$$

Cosine funcⁿ:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (ii)}$$

Successive integral: - [SLATE]

$$y = \int u \cdot v dx = uv_1 - v_1'v_2 + v_2''v_3 - v_3'''v_4 + \dots$$

where,

$u \rightarrow$ derivative

$v \rightarrow$ integration

Complex Fourier series

We have Fourier series of $f(x)$ with Period 2π is

$$f(x) = a_0 + \sum_{n=-\infty}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (iii)}$$

We can modify the Fourier series into complex form as.

$$e^{inx} = \cos nx + i \sin nx$$

$$e^{-inx} = \cos nx - i \sin nx$$

This gives.

$$\cos nx = \frac{1}{2} (e^{inx} + e^{-inx})$$

$$\sin nx = \frac{1}{2i} (e^{inx} - e^{-inx})$$

$$\text{Hence, } a_n \cos nx + b_n \sin nx = a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) +$$

$$\frac{1}{2i} b_n (e^{inx} - e^{-inx})$$

After solving that,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad \text{--- (A)}$$

$$\text{Where } C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$n = 0, \pm 1, \pm 2, \dots$

Where (A) is the complex form of fourier series

If interval (period) is $2L$:-

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/L}$$

$$C_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx$$

exam

Q. Find the complex fourier series of $f(x) = e^x$ in $-\pi < x < \pi$.

$\Rightarrow$ We have,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad \text{--- (i)}$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x \cdot e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x(1-in)} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{x(1-in)}}{(1-in)} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-in)} \left[e^{\pi(1-in)} - e^{-\pi(1-in)} \right]$$

$$= \frac{1}{\pi(1-in)} \left[\frac{e^{\pi(1-in)} - e^{-\pi(1-in)}}{2} \right] \quad \text{--- (ii)}$$

$$= \frac{1}{\pi(1-in)} \left[\frac{e^{\pi}(\cos n\pi + i \sin n\pi) - e^{-\pi}(\cos n\pi - i \sin n\pi)}{2} \right]$$

here,

$$e^{\pi - in\pi} = e^{\pi} [\cos n\pi - i \sin n\pi] = e^{\pi} (-1)^n$$

$$e^{-\pi + in\pi} = e^{-\pi} e^{in\pi} = e^{-\pi} (-1)^n$$

$$\begin{aligned}
 \text{Now,} \\
 &= \frac{1}{\pi(1-in)} \left[\frac{e^{i\pi}(-1)^n - e^{-i\pi}(-1)^n}{2} \right] \\
 &= \frac{(-1)^n}{\pi(1-in)} \left[\frac{e^{i\pi} - e^{-i\pi}}{2} \right] \\
 &= \frac{(-1)^n}{\pi(1-in)} \times \frac{(1+in)}{(1+in)} \sinh \pi \quad (\sinh \pi) \\
 &= \frac{\sinh \pi}{\pi} \frac{(-1)^n (1+in)}{(1+n^2)}
 \end{aligned}$$

Now, from eqn (1),

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1+in) \sinh \pi}{\pi(1+n^2)} e^{ina}$$

Fourier integral:

Let $f(x)$ be a funcⁿ of period $2L$
then Fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}) \quad (1)$$

where

$$w_n = \frac{n\pi}{L}$$

But, The Fourier integral of $f(x)$ is:

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw$$

Where,

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos wt \, dt$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin wt \, dt$$

Q. Find the Fourier integral of $f(x) = \begin{cases} \frac{1}{x} & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

⇒ Given, Fourier integral of $f(x)$ is

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw \quad (1)$$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos wt \, dt$$

$$= \frac{1}{\pi} \int_{-1}^1 \frac{1}{t} \cos wt \, dt = \frac{1}{\pi} \left[\frac{\sin wt}{w} \right]_{-1}^1$$

$$= \frac{1}{\omega\pi} [\sin\omega - (\sin(-\omega))] \checkmark$$

$$= \frac{2\sin\omega}{\omega\pi} \checkmark$$

Again.

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin\omega t dt$$

$$= \frac{1}{\pi} \int_{-1}^1 1 \cdot \sin\omega t dt$$

$$= \frac{1}{\pi} \left[-\frac{\cos\omega t}{\omega} \right]_{-1}^1 = 0 \quad [\text{odd func}^n] \checkmark$$

$$= \frac{1}{\pi\omega} [-\cos\omega - (-\cos\omega)] = 0$$

Now, from (i),

$$f(x) = \int_0^{\infty} \frac{2\sin\omega}{\omega\pi} \cos\omega x d\omega \checkmark$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

Formula. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$

3. Fourier Sine / Cosine Integral

(a) Fourier Cosine integral

$$f(x) = \int_0^\infty A(\omega) \cos \omega x \, d\omega \quad \text{--- (a)}$$

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \cos \omega t \, dt \quad \text{--- (b)}$$

(b) Fourier Sine integral

$$f(x) = \int_0^\infty B(\omega) \sin \omega x \, d\omega \quad \text{--- (c)}$$

$$B(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \sin \omega t \, dt \quad \text{--- (d)}$$

e.g Q. Find the Cosine integral of $f(x) = e^{-x} + e^{-2x}$ for $x > 0$;

$\Rightarrow$ Here, Fourier Cosine integral,

$$f(x) = \int_0^\infty A(\omega) \cos \omega x \, d\omega \quad \text{--- (i)}$$

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \cos \omega t \, dt$$

$$e^{-\infty} = 0$$

$$e^{\infty} = \infty$$

then,

$$A(\omega) = \frac{2}{\pi} \int_0^\infty (e^{-t} + e^{-2t}) \cos \omega t \, dt$$

$$= \frac{2}{\pi} \left[\int_0^\infty e^{-t} \cos \omega t \, dt + \int_0^\infty e^{-2t} \cos \omega t \, dt \right]$$

$$= \frac{2}{\pi} \left[\frac{e^{-t}}{1 + \omega^2} (-\cos \omega t + \omega \sin \omega t) \right]_0^\infty + \left[\frac{e^{-2t}}{4 + \omega^2} (-2 \cos \omega t + \omega \sin \omega t) \right]_0^\infty$$

$$= \frac{2}{\pi} \left[0 - \frac{1}{1 + \omega^2} (-1) + 0 - \frac{1}{4 + \omega^2} (-2) \right]$$

$$= \frac{2}{\pi} \left[\frac{1}{1 + \omega^2} + \frac{2}{4 + \omega^2} \right]$$

from (i) becomes,

$$f(x) = \frac{2}{\pi} \int_0^\infty \left[\frac{1}{1 + \omega^2} + \frac{2}{4 + \omega^2} \right] \cos \omega x \, d\omega$$

P.O

Prove that, $\int_0^{\infty} \frac{\sin w \cos wx}{w} dw = \begin{cases} \frac{\pi}{2} & \text{if } 0 \leq x < 1 \\ \frac{\pi}{4} & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$

$\Rightarrow$ we have to show,

$$f(x) = \begin{cases} \frac{\pi}{2} & \text{if } 0 \leq x < 1 \\ \frac{\pi}{4} & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases} = \int_0^{\infty} \frac{\sin w \cos wx}{w} dw.$$

then fourier cosine integral

$$f(x) = \int_0^{\infty} A(w) \cos wx dw \quad \text{--- (1)}$$

$$\text{Where, } A(w) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos wt dt$$

$$= \frac{2}{\pi} \int_0^1 \frac{\pi}{2} \cos wt dt$$

$$= \left[\frac{\sin wt}{w} \right]_0^1$$

$$= \frac{\sin w}{w}$$

Hence $f(x)$ becomes,

$$f(x) = \int_0^{\infty} \frac{\sin w \cos wx}{w} dw$$

proved,

Q. show that, (i) $\int_0^{\infty} \left(\frac{1 - \cos \pi w}{w} \right) \sin xw dw = \begin{cases} \frac{\pi}{2} & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

$\Rightarrow$ here, fourier sine integral,

$$f(x) = \int_0^{\infty} B(w) \sin wx dw \quad \text{--- (2)}$$

$$B(w) = \frac{2}{\pi} \int_0^{\infty} f(t) \sin wt dt$$

$$= \frac{2}{\pi} \int_0^1 \frac{\pi}{2} \sin wt dt$$

$$= \left[-\frac{\cos wt}{w} \right]_0^1 = \left[-\frac{\cos w \pi}{w} + \frac{1}{w} \right] = \left[\frac{1 - \cos \pi w}{w} \right]$$

Now,

(i) becomes, $f(x) = \int_0^{\infty} \left(\frac{1 - \cos \pi w}{w} \right) \sin xw dw$ proved. #

Note: Use combine formula, ✓

$$\textcircled{2} \int_0^{\infty} \left(\frac{\cos \omega x + \omega \sin \omega x}{1 + \omega^2} \right) d\omega = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

⇒ Using both Sine and Cosine integral,

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x d\omega \quad \text{--- (A)}$$

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos \omega t dt.$$

$$= \frac{2}{\pi} \int_0^{\infty} \pi e^{-t} \cos \omega t dt.$$

$$= 2 \left[\frac{e^{-t}}{1 + \omega^2} (-1) \cos \omega t + \omega \sin \omega t \right]_0^{\infty}$$

$$= 2 \left[0 - \frac{1}{1 + \omega^2} (-1) \right]$$

$$= \frac{2}{1 + \omega^2}$$

$$\text{Again, } f(x) = \int_0^{\infty} B(\omega) \sin \omega x d\omega \quad \text{--- (B)}$$

$$B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \sin \omega t dt$$

$$= \frac{2}{\pi} \int_0^{\infty} \pi e^{-t} \sin \omega t dt$$

$$= 2 \left[\frac{e^{-t}}{1 + \omega^2} (-1) \sin \omega t - \omega \cos \omega t \right]_0^{\infty}$$

$$= 2 \left[0 - \frac{1}{1 + \omega^2} (-\omega) \right]$$

$$= \frac{2\omega}{1 + \omega^2}$$

Now, both Cosine and Sine integral, becomes,

$$f(x) = \int_0^{\infty} \left[\frac{2}{1 + \omega^2} \cos \omega x + \frac{2\omega}{1 + \omega^2} \sin \omega x \right] d\omega$$

$$= \int_0^{\infty} \left(\frac{\cos \omega x + \omega \sin \omega x}{1 + \omega^2} \right) d\omega$$

proved

J.M.F

Fourier Cosine and Sine Transform:
We have the Fourier cosine integral

of $f(x)$ is:
$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x d\omega \quad \text{--- (i)}$$

where, $A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos \omega t dt \quad \text{--- (ii)}$

Now, we set.

$$A(\omega) = \sqrt{\frac{2}{\pi}} \left[\int_0^{\infty} \sqrt{\frac{2}{\pi}} f(t) \cos \omega t dt \right]$$

$$= \sqrt{\frac{2}{\pi}} f_c^{\wedge}(\omega)$$

Hence,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c^{\wedge}(\omega) \cos \omega x d\omega \quad \text{--- (iii)}$$

Where,

$$f_c^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t dt. \quad \text{--- (iv)}$$

is said to be Fourier Cosine Transform of $f(x)$.

Fourier Sine transform :-

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s^{\wedge}(\omega) \sin \omega x d\omega \quad \text{--- (v)}$$

$$f_s^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t dt \quad \text{--- (vi)}$$

eg. Q. Find the Fourier cosine transform of $f(x) = e^{-mx}$, $m > 0$.

$\Rightarrow$ We have.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c^{\wedge}(\omega) \cos \omega x d\omega \quad \text{--- (i)}$$

where,

$$f_c^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-mt} \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-mt}}{m^2 + \omega^2} (-m \cos \omega t + \omega \sin \omega t) dt$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{m^2 + \omega^2} (-m)t \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{m}{m^2 + \omega^2} \right]$$

Q. Find the cosine transform of $f(x) = \begin{cases} x^2 & \text{for } 0 < x < \infty \\ 0 & \text{for } x > \infty \end{cases}$

$\Rightarrow$ We have.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c^{\wedge}(\omega) \cos \omega x d\omega \quad \text{--- (i)}$$

where,

$$f_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^t t^2 \cos \omega t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[t^2 \frac{\sin \omega t}{\omega} - 2t \left(\frac{\cos \omega t}{\omega^2} \right) + 2 \left(\frac{\sin \omega t}{\omega^3} \right) \right]_0^t$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1 \cdot \sin \omega}{\omega} + \frac{2 \cdot 1 \cdot \cos \omega}{\omega^2} - \frac{2 \sin \omega}{\omega^3} \right] - (0 + 0 - 0)$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega}{\omega} + \frac{2 \cos \omega}{\omega^2} - \frac{2 \sin \omega}{\omega^3} \right]$$

Q. Find the Fourier sine and cosine transform.
 $f(x) = 2e^{-5x} + 5e^{-2x}$.

⇒ We have, for cosine,

$$f_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (2e^{-5t} + 5e^{-2t}) \cos \omega t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^{\infty} 2e^{-5t} \cos \omega t \, dt + \int_0^{\infty} 5e^{-2t} \cos \omega t \, dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[2 \left(\frac{e^{-5t}}{25 + \omega^2} (5 \cos \omega t + \omega \sin \omega t) \right) \Big|_0^{\infty} + 5 \left(\frac{e^{-2t}}{4 + \omega^2} (-2 \cos \omega t + \omega \sin \omega t) \right) \Big|_0^{\infty} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[2 \left(0 - \frac{e^{-5}}{25 + \omega^2} (-5) \right) + 5 \left(0 - \frac{e^{-2}}{4 + \omega^2} (-2) \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{10e^{-5}}{25 + \omega^2} + \frac{10e^{-2}}{4 + \omega^2} \right]$$

for sine,

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Note: The Fourier sine and cosine transform are $f_s(f) = f_s^n$ and $f_c(f) = f_c^n$ then inverse of sine and cosine are denoted as f_s^{-1} and f_c^{-1} respectively.

Theorem: Let $f_s(f)$ and $f_c(g)$ are Fourier cosine and sine transform of $f(x)$ and $g(x)$. Also a and b are constant then
 $f_c[af + bg] = a f_c(f) + b f_c(g)$

Theorem: Let $f(x)$ be continuous and absolutely integrable on x -axis, let $f'(x)$ be piecewise continuous on each finite interval and let $f(x) \rightarrow 0$ as $x \rightarrow \infty$, then

$$f_s(f'(x)) = \omega f_c(f(x)) - \sqrt{\frac{2}{\pi}} f(0)$$

and

$$f_c(f'(x)) = -\omega f_s(f(x))$$

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Q. Find the Fourier transform of $f(x) = e^{-ax^2}$ for $a > 0$.

Q. Find the Fourier cosine transform of $f(x) = e^{-x^2}$.
 $\Rightarrow$ By definition of cosine transform of e^{-x^2}

$$f_c(e^{-x^2}) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t^2} \cos \omega t dt = I \text{ say.}$$

$$I = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t^2} \cos \omega t dt \dots \dots \text{--- (1)}$$

Differentiate both sides w.r. to ' ω ' we get.

$$\frac{dI}{d\omega} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t^2} \frac{d(\cos \omega t)}{d\omega} dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t^2} (-t \sin \omega t) dt$$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \{(-2t) e^{-t^2}\} \sin \omega t dt$$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left[\sin \omega t \right]_0^{\infty} - \omega \left[\cos \omega t e^{-t^2} dt \right]_0^{\infty}$$

$$= -\frac{\omega}{2} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos \omega t e^{-t^2} dt$$

$$\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \frac{dI}{d\omega} = -\frac{\omega}{2} I$$

$$\Rightarrow \frac{dI}{I} = -\frac{\omega}{2} d\omega$$

Integrating,

$$\ln I = -\frac{\omega^2}{2} + \ln(A)$$

$$I = A e^{-\frac{\omega^2}{2}} \dots (*)$$

Put $\omega=0$
 $I = A$ then,

$$I = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t^2} dt \quad [\text{by eqn } (i)]$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow A = I = \frac{1}{\sqrt{2}}$$

So, $I = \int_0^{\infty} e^{-x^2} = \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{2}} \quad \#$

$$\begin{aligned} \int u \cdot v dx &= u \int v dx - \int \left[\frac{du}{dx} \int v dx \right] dx \\ &= \sin \omega t \int d(e^{-t^2}) - \int \left[\frac{d(\sin \omega t)}{dt} \int d(e^{-t^2}) \right] dt \\ &= \sin \omega t \cdot e^{-t^2} - \int \omega \cos \omega t \cdot e^{-t^2} dt \end{aligned}$$

V.V.I

Complex form of Fourier transform.

Complex Fourier integral of $f(x)$:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\omega(x-t)} dt d\omega$$

Fourier transform and its inverse.

Now,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{i\omega(x-t)} dt \right] d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right] e^{i\omega x} d\omega$$

Put $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \dots (i)$

then,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \dots (ii)$$

Hence, (i) is Fourier transform of $f(x)$ and (ii) is inverse Fourier transform of $\hat{f}(\omega)$.

Example:-
find the complex fourier transform of

$$f(x) = \begin{cases} kx & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ By definition, complex fourier transform of $f(x)$ is

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b k t e^{-i\omega t} dt$$

$$= \frac{k}{\sqrt{2\pi}} \int_a^b t e^{-i\omega t} dt$$

$$= \frac{k}{\sqrt{2\pi}} \left[t \frac{e^{-i\omega t}}{(-i\omega)} - \frac{e^{-i\omega t}}{(-i\omega)^2} \right]_a^b$$

$$= \frac{k}{\sqrt{2\pi}} \left[\frac{b e^{-i\omega b}}{i\omega} - \frac{e^{-i\omega b}}{(-\omega^2)} - \left(\frac{a e^{-i\omega a}}{(-i\omega)} - \frac{e^{-i\omega a}}{(-\omega^2)} \right) \right]$$

$$= \frac{k}{\sqrt{2\pi}} \left[-\frac{b e^{-i\omega b}}{i\omega} + \frac{e^{-i\omega b}}{\omega^2} + \frac{a e^{-i\omega a}}{i\omega} + \frac{e^{-i\omega a}}{\omega^2} \right]$$

$$= \frac{k}{\sqrt{2\pi}} \left[\frac{e^{-i\omega b} + e^{-i\omega a}}{\omega^2} + \left(\frac{a e^{-i\omega a}}{i\omega} - \frac{b e^{-i\omega b}}{i\omega} \right) \right]$$

$$= \frac{k}{\sqrt{2\pi}} \left[\frac{e^{-i\omega b} + e^{-i\omega a}}{\omega^2} - i \left(\frac{a e^{-i\omega a}}{\omega} - \frac{b e^{-i\omega b}}{\omega} \right) \right]$$

Prm 5: Exam

Q. find the complex fourier transform of $f(x) = e^{ax}$

Note: linearity theorem of fourier transform.

Let $f(x)$ and $g(x)$ are functions, whose fourier transform exists and for any constant a and b .

$$f[af(x) + bg(x)] = a f[f(x)] + b f[g(x)]$$

2. Let $f(x)$ be cont. on x -axis and $f(x) \rightarrow 0$ as $(x) \rightarrow \infty$ and $f(x)$ exists (and integrable) then the following condition holds.

$$f[f'(x)] = i\omega f[f(x)]$$

$$f[f''(x)] = -\omega^2 f[f(x)]$$

Example: find the fourier transform of $\sin 2x$

$\Rightarrow$ Since, $f(x) = \sin 2x$

$$f'(x) = 2 \cos 2x$$

$$f''(x) = -4 \sin 2x$$

$$f'''(x) = -4 f(x)$$

∴ taking Fourier transform on both sides

$$f[f''(x)] = -4 f[f(x)]$$

$$-w^2 f[f(w)] = -4 f[f(x)]$$

$$f[f(w)](-w^2 + 4) = 0$$

$$f[f(x)] = 0$$

$$f[\sin 2x] = 0.$$

#

State Convolution theorem :-

The convolution theorem of two function $f(x)$ and $g(x)$ is denoted by $f * g$ and defined by.

$$f * g = \int_{-\infty}^{\infty} f(x-p) g(p) dp = \int_{-\infty}^{\infty} f(p) g(x-p) dp$$

Parseval's identity for Fourier Transform.
If $f(x)$ and $g(x)$ two piecewise continuous function then,

$$\int_{-\infty}^{\infty} f(w) \bar{g}(w) dw = \int_{-\infty}^{\infty} f(x) \bar{g}(x) dx$$

and,

$$\int_{-\infty}^{\infty} |f(w)|^2 dw = \int_{-\infty}^{\infty} |f(x)|^2 dx \text{ where bar indicate the complex conjugate.}$$

Parseval's identity on Cosine transform.

$$\int_0^{\infty} f_c(w) g_c(w) dw = \int_0^{\infty} f(x) g(x) dx$$

and

$$\int_0^{\infty} |f_c(w)|^2 dw = \int_0^{\infty} |f(x)|^2 dx$$

Parseval's identity for Sine transform.

$$\int_0^{\infty} f_s(w) g_s(w) dw = \int_0^{\infty} f(x) g(x) dx$$

and,

$$\int_0^{\infty} |f_s(w)|^2 dw = \int_0^{\infty} |f(x)|^2 dx$$

Q. Find sine and cosine transform form..

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < L \\ 0 & \text{for } x \geq L. \end{cases} \text{ Show that (a) } \int_0^\infty \left(1 - \frac{\cos x}{x}\right)^2 dx = \frac{\pi}{2}$$

$$(b) \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$

$\Rightarrow$ We have. Fourier sine transform of $f(x)$.

$$(a) f_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^L 1 \cdot \sin \omega t dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos \omega}{\omega} \right]_0^L$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos \omega}{\omega} + \frac{1}{\omega} \right]$$

$$= \frac{1}{\omega} \sqrt{\frac{2}{\pi}} (1 - \cos \omega)$$

Again, Fourier cosine transform of $f(x)$ is,

$$(b) f_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t dt = \sqrt{\frac{2}{\pi}} \int_0^L 1 \cdot \cos \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega}{\omega} \right]_0^L$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega}{\omega} \right]$$

Now, we know that, By Parseval's identity for sine transform.

$$\int_0^\infty |f_s(\omega)|^2 d\omega = \int_0^\infty |f_c(x)|^2 dx$$

$$\Rightarrow \int_0^\infty \left| \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos \omega}{\omega} \right] \right|^2 d\omega = \int_0^L |1|^2 dx$$

$$\Rightarrow \int_0^\infty \frac{2}{\pi} \left(\frac{1 - \cos \omega}{\omega} \right)^2 d\omega = \left[x \right]_0^L$$

$$= \int_0^\infty \left(\frac{1 - \cos \omega}{\omega} \right)^2 d\omega = \frac{\pi}{2}$$

This gives, substituting $\omega = x$. then

$$\int_0^\infty \left(\frac{1 - \cos x}{x} \right)^2 dx = \frac{\pi}{2} \text{ proved.}$$

Again, By Parseval's identity for cosine transform

$$\int_0^{\infty} |f(\omega)|^2 d\omega = \int_0^L |f(x)|^2 dx$$

$$\Rightarrow \int_0^{\infty} \left| \sqrt{\frac{2}{\pi}} \left(\frac{\sin \omega}{\omega} \right) \right|^2 d\omega = \int_0^L (1)^2 dx.$$

$$\Rightarrow \int_0^{\infty} \frac{2}{\pi} \left(\frac{\sin \omega}{\omega} \right)^2 d\omega = \left[x \right]_0^L.$$

$$\Rightarrow \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin \omega}{\omega} \right)^2 d\omega = L.$$

$$\int_0^{\infty} \left(\frac{\sin \omega}{\omega} \right)^2 d\omega = \pi/2$$

replace $\omega = x$ then,

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \pi/2 \quad \text{proved} \quad \#$$

$$\int e^{ax} \cos bx = \left[\frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right]$$

1. find the fourier cosines transform of $f(x) = e^{-mx}$ for $m > 0$ and show that:

$$\int_0^{\infty} \frac{\cos kx}{1+x^2} dx = \frac{\pi}{2} e^{-k}$$

$\Rightarrow$ we have fourier cosine transform.

$$f(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t dt.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-mt} \cos \omega t dt.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-mt}}{m^2 + \omega^2} (-m \cos \omega t + \omega \sin \omega t) \right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{d}{m^2 + \omega^2} - (-m) \right] \quad \left[\frac{d}{m^2 + \omega^2} - (-m) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{m}{m^2 + \omega^2} \right].$$

Now by Parseval's identity for cosine transform.

2. find the fourier transform
 $f(x) = e^{-x^2/2}$.

$$\int_{-\infty}^{\infty} f(\omega) G_2(\omega) d\omega = \int_{-\infty}^{\infty} f(x) g(x) dx.$$

by inverse formula,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2m}{\pi}} \left(\frac{2m}{m^2 + \omega^2} \right) \cos \omega x d\omega$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{m}{m^2 + \omega^2} \right) \cos \omega x d\omega.$$

$$= \frac{2m}{\pi} \int_0^{\infty} \frac{\cos \omega x}{m^2 + \omega^2} d\omega.$$

$$e^{-mx} = \frac{2m}{\pi} \int_0^{\infty} \frac{\cos \omega x}{m^2 + \omega^2} d\omega$$

Setting $m=1$ $\omega=x \Rightarrow x=k$.

$$\frac{1}{2} e^{-kx} = \int_0^{\infty} \frac{\cos kx}{1+x^2} dx$$

$$e^{-x^2/2} \Rightarrow \frac{1}{\sqrt{2}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2}}$$

2. Here $f(x) = e^{-x^2/2} \Rightarrow f'(x) = -\frac{1}{2} e^{-x^2/2} \cdot x$
 Taking fourier transform, on both side.

$$f[f'(x)] = f\left[-\frac{x}{2} e^{-x^2/2}\right]$$

$$f[f'(x)] = f\left[-\frac{x}{2} f(x)\right]$$

$$2 f[f'(x)] = f[-x e^{-x^2/2}]$$

$$2ix f[f'(x)] = f\left[-x e^{-x^2/2}\right]$$

Ex. 7.3]

(13) Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$

for $x > 0, a > 0$. then show that $\int_0^\infty \frac{e^{-at}}{t} \sin \omega t dt = \tan^{-1}\left(\frac{\omega}{a}\right)$
 $\Rightarrow$ here $f(x) = \frac{e^{-ax}}{x}$

We have, The sine Fourier transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-at}}{t} \sin \omega t dt. \quad \text{--- (i)}$$

Let
$$I = \int_0^\infty \frac{e^{-at}}{t} \sin \omega t dt$$

Diff I w.r. to 'w' we get,

$$\frac{dI}{d\omega} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-at}}{t} \times \frac{d(\sin \omega t)}{d\omega} \times dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-at}}{t} \cdot \cos \omega t \times t dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \cos \omega t dt.$$

$$= \left[\frac{e^{-at}}{a^2 + \omega^2} (-a \cos \omega t + \omega \sin \omega t) \right]_0^\infty$$

$$= \left[0 - \left(\frac{e^0}{a^2 + \omega^2} (-a) \right) \right]$$

$$\frac{dI}{d\omega} = \left(\frac{a}{a^2 + \omega^2} \right) \sqrt{\frac{2}{\pi}}$$

$$dI = \left(\frac{a}{a^2 + \omega^2} \right) d\omega$$

integrating, we get,

$$I = a \int \frac{d\omega}{a^2 + \omega^2} = \sqrt{\frac{2}{\pi}} \frac{a}{a} \tan^{-1}\left(\frac{\omega}{a}\right)$$

Hence,
$$f(\omega) = \int_0^\infty \frac{e^{-at}}{t} \sin \omega t dt = \tan^{-1}\left(\frac{\omega}{a}\right) \cdot \sqrt{\frac{2}{\pi}}$$

inverse Fourier sine function is,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\omega) \sin \omega x d\omega$$

$$\frac{e^{-ax}}{x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\omega}{a}\right) \cdot \sin \omega x d\omega$$

Substitute, $x = t$, and $u = x$.

$$\frac{e^{-a}}{1} = \frac{2}{\pi} \int_0^{\infty} \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx$$

$$\Rightarrow \int_0^{\infty} \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx = \frac{\pi}{2} e^{-a} \quad \text{proved,}$$

Q.E.D.

Partial Differential Equation [8+7=15M]

↳ A equation involving one or more partial derivatives of an (unknown) function of two or more independent variables is called partial diff eqⁿ.

For Example: one dimensional wave eqⁿ: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

one-D heat eqⁿ: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Two-D Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2-D poisson eqⁿ: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

X 2-D wave eqⁿ: $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

X 2-D Laplace eqⁿ: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

2-D ~~heat~~ heat eqⁿ: $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

Where c is constant t is time period and x and y are variable

Diff. eqⁿ. first

→ $\frac{dy}{dx} + Py = Q$ (non-homo. first order diff. eqⁿ)

$$y \times I.f = \int (Q \times I.f) dx + C \quad \hookrightarrow \quad I.f = e^{\int P dx}$$

→ $M dx + N dy = 0$ → Exact diff. eqⁿ.

$$\int M dx + \int (\text{term of } N \text{ free from } x) dy = C$$

Second order diff. eqⁿ:-

Homogeneous:-

(i) If roots are real and distinct

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} \quad \text{eg. } y'' - 4y = 0$$

Auxiliary eqⁿ:

$$m^2 - 4 = 0$$

$$m = 2, -2$$

(ii) If roots are equals.

$$y = (C_1 + C_2 x) e^{mx}$$

$$\text{eg. } y'' - 2y' + y = 0$$

$$\text{or, } m^2 - 2m + 1 = 0$$

$$\text{or, } (m-1)^2 = 0$$

$$m = 1, 1$$

$$y = (C_1 + C_2 x) e^x$$

③ If Complex root: $y = e^{px} (A \cos qx + B \sin qx)$

eg. $y'' + 4y = 0$
 $m^2 - (2i)^2 = 0$

$(m+2i)(m-2i) = 0$
 $m = \pm 2i$

$y = e^{0x} (A \cos 2x + B \sin 2x)$
 $= A \cos 2x + B \sin 2x$

Solve: $U_2x + 9U = 0$

Making aux. eqn of given eqn.
 $m^2 + 9 = 0$

$m = \pm 3i = 0 \pm 3i = p + iq$

$y = A \cos 3x + B \sin 3x$

Solve: $U_{xy} = U_x$

$\Rightarrow$ Given $U_{xy} = U_x$

Put $U_x = k$

$K_y =$

$y = e^{(y+c)} = e^y \cdot e^c$

$K_y = k, \frac{\partial K}{\partial y} = k$

$\frac{\partial K}{\partial y} = k$
 integrate.

$\log K = y + c, \quad c \text{ is constant,}$

$K = e^y \cdot e^c$

$\frac{\partial K}{\partial x} = e^c e^y$

$\partial K = e^c e^y dx$
 on integrating,

$K = e^y \int e^c dx + D$ Where D is const.

then the required soln is,

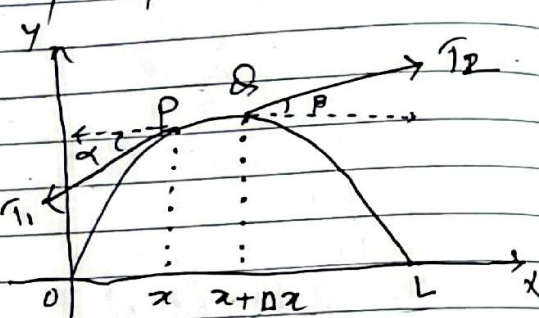
$U = e^y \int e^c dx + D$

V.V.S.M.P

State and prove / Derive one D-Wave eqⁿ (8M)

→ Let us suppose that the force acting on small portion of strings. Since the strings does not offer to bend, the tension caused is tangential to the curve of the string at each point.

Let T_1 and T_2 are tension at the ends points P & Q of the portion. Since, the points of the string move vertically, so there is no motion horizontally direction. Therefore the horizontal components of the tension must be constant.



i.e. $T_1 \cos \alpha = T_2 \cos \beta = T$ (say) = constant (i)

But in the vertical direction, forces on vertical component are $-T_1 \sin \alpha$ and $T_2 \sin \beta$ of T_1 and T_2 . Here, -ve sign means component of P is directed down wards. Then by Newton's second law, we get,

The resultant force is equal with the mass of the portion times the acceleration.

i.e. $T_2 \sin \beta - T_1 \sin \alpha = \Delta x \times \frac{\partial^2 y}{\partial t^2}$ (ii)

Where Δ be the mass of undisturbed string per unit length, Δx is the length of the portion of the undisturbed string.

Dividing eqⁿ (ii) by (i), we get.

$$\frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \frac{\Delta x}{T} \cdot \frac{\partial^2 y}{\partial t^2}$$

$$\tan \beta - \tan \alpha = \frac{\Delta x}{T} \cdot \frac{\partial^2 y}{\partial t^2} \quad \text{(iii)}$$

Since, $\tan \beta$ is a slope at Q and $\tan \alpha$ is the slope at P. as,

$$\tan \alpha = \left. \frac{\partial y}{\partial x} \right|_{x}$$

$$\tan \beta = \left. \frac{\partial y}{\partial x} \right|_{x+\Delta x}$$

from (iii), we get,

$$\left[\left. \frac{\partial y}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial y}{\partial x} \right|_x \right] = \frac{\Delta x}{T} \cdot \frac{\partial^2 y}{\partial t^2}$$

Dividing both side by Δx and taking
 $\lim_{\Delta x \rightarrow 0}$ then,

$$\lim_{\Delta x \rightarrow 0} \left[\frac{\left(\frac{\partial u}{\partial x} \right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x} \right)_x}{\Delta x} \right] = \frac{1}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \left[\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \right], \text{ Where } c^2 = \frac{T}{\rho}$$

This is the required one D - wave eqn.

Solution of one-dim. wave eqn. under certain initial and Boundary condn.

$$\Rightarrow \text{The one-dim. wave eqn is,} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (i)}$$

Where, $y(x,t)$ is the deflection of string with boundary condn,

$$u(0,t) = 0 \text{ and } u(L,t) = 0 \text{ for all } t \quad \text{--- (ii)}$$

and initial condn

$$u(x,0) = f(x) \quad \text{(initial deflection)} \quad \text{--- (iii)}$$

$$\frac{\partial u}{\partial t} \Big|_{t=0} = g(x) \quad \text{(initial velocity)} \quad \text{--- (iv)}$$

$$\text{let } u(x,t) = f(x) G(t) \quad \text{--- (v)}$$

Diff. eqn (v) partially w.r.t t and x , we get

$$\frac{\partial u}{\partial x} = f'(x) G \Rightarrow \frac{\partial^2 u}{\partial x^2} = f''(x) G$$

$$\frac{\partial u}{\partial t} = F G' \Rightarrow \frac{\partial^2 u}{\partial t^2} = F G''$$

Then eqn (i) becomes,

$$F G'' = c^2 F'' G$$

$$\Rightarrow \frac{G''}{c^2 G} = \frac{F''}{F} = k \text{ (say)}$$

This gives,

$$(\because \sin \theta = 0 \Rightarrow \theta = n\pi)$$

$$\ddot{G} - c^2 k G = 0 \quad \text{--- (vii)}$$

$$F'' - k F = 0 \quad \text{--- (viii)}$$

Which are ordinary diff. eqⁿ.

Now, we have find the value of F and G that depend on the eqⁿ (vi) and (vii) with boundary conditions.

After solving the eqⁿ (vi) we get, $F = A e^{kx} + B e^{-kx}$
again if we put k is positive, let $k = \lambda^2$
it remains.

$$F = A e^{\lambda x} + B e^{-\lambda x}$$

if taking k is (-ve) as $k = -p^2$ we get,

$$F'' + p^2 F = 0 \quad \text{[by (b)]}$$

$$m^2 + p^2 = 0$$

$$m^2 = -p^2$$

$$F = A \cos px + B \sin px$$

$$\text{We have, } F(0) = A = 0$$

$$F(L) = B \sin pL = 0$$

$$\sin pL = 0 \quad \text{where } B \neq 0.$$

$$pL = n\pi$$

$$p = \frac{n\pi}{L} \quad \text{for } n \text{ is a integer.}$$

We have $B \neq 0$, setting $B=1$ we get

$$f(x) = \sin \frac{n\pi}{L} x.$$

We obtain infinitely many solutions.

$$f(x) = f_n(x), \text{ thus.}$$

$$f_n(x) = \sin \frac{n\pi}{L} x \quad \text{for } n = 1, 2, 3, \dots \quad \text{--- (4)}$$

Solving eqⁿ (vii),

We have, $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$, then
eqⁿ (vii) is.

$$\ddot{G} - c^2 k G = 0$$

$$\Rightarrow \ddot{G} + \lambda_n^2 G = 0 \quad \text{where, } \lambda_n = \frac{cn\pi}{L}$$

Its solution, $G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t$

Therefore, The required solution of eqⁿ (i) is

$$U(x, t) = f(x). G(t).$$

$$U_n(x, t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \frac{\sin n\pi x}{L} \quad \text{--- (A)}$$

for $n = 1, 2, 3, \dots$

Where. The set of $(\lambda_1, \lambda_2, \lambda_3, \dots)$ is called the spectrum of the vibrating strings. and its value

$$\lambda_n = \frac{n\pi}{L} \quad \#$$

Hence, the total solution of the one dim. wave eqⁿ is,

$$U(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \frac{\sin n\pi x}{L}$$

Where,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx.$$

$$B_n^* = \frac{2}{L\lambda_n} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$$

where $f(x)$ is initial deflection and $g(x)$ is initial velocity.

P.V.

initial deflection - $f(x)$
initial velocity - $g(x)$
displacement = $U(x, t)$

A tight string of length 20cm fastened at both ends is displaced from its position of equilibrium, by imparting to each of its point an initial velocity $g(x) = x$ for $0 \leq x \leq 10$

Where x being distance from end. find displacement at any time.

$\Rightarrow$ The required displacement of wave eqⁿ is.

$$U(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \frac{\sin n\pi x}{L} \quad \text{--- (1)}$$

Where,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

$$B_n^* = \frac{2}{L\lambda_n} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx, \quad \lambda_n = \frac{n\pi}{L}$$

Since, $f(x) = 0 \Rightarrow B_n = 0$ then, we find

$$B_n^* \quad \lambda_n = \frac{n\pi}{L}$$

$$B_n^* = \frac{2}{20\lambda_n} \left[\int_0^{10} x \cdot \sin \frac{n\pi}{20} x \, dx + \int_{10}^{20} (20-x) \sin \frac{n\pi}{20} x \, dx \right]$$

$$= \frac{2}{c\pi h} \left[\frac{x \cdot (-\cos \frac{n\pi}{20} x)}{\left(\frac{n\pi}{20}\right)} - \frac{1 \cdot (-\sin \frac{n\pi}{20} x)}{\left(\frac{n\pi}{20}\right)^2} \right]_0^{10}$$

$$\left[\frac{(20-x) \cdot (-\cos \frac{n\pi}{20} x)}{\left(\frac{n\pi}{20}\right)} - \frac{(-1) \cdot (-\sin \frac{n\pi}{20} x)}{\left(\frac{n\pi}{20}\right)^2} \right]_{10}^{20}$$

$$= \frac{2}{c\pi h} \left[\frac{-10 \cos \frac{n\pi}{2}}{\left(\frac{n\pi}{20}\right)} + \frac{\sin \frac{n\pi}{2}}{\left(\frac{n\pi}{20}\right)^2} \right] + 0 - \left[\frac{-10 \cos \frac{n\pi}{2}}{\left(\frac{n\pi}{20}\right)} \right]$$

$$\left[\frac{-\sin \frac{n\pi}{2}}{\left(\frac{n\pi}{20}\right)^2} \right]$$

$$= \frac{2}{c\pi h} \left[\frac{-10 \cos \frac{n\pi}{2}}{\left(\frac{n\pi}{20}\right)} + \frac{\sin \frac{n\pi}{2}}{\left(\frac{n\pi}{20}\right)^2} + \frac{10 \cos \frac{n\pi}{2}}{\left(\frac{n\pi}{20}\right)} + \frac{\sin \frac{n\pi}{2}}{\left(\frac{n\pi}{20}\right)^2} \right]$$

$$= \frac{2}{c\pi h} \left[\frac{400 \sin \frac{n\pi}{2}}{h^2 \pi^2} + \frac{400 \sin \frac{n\pi}{2}}{n^2 \pi^2} \right]$$

$$= \frac{1600}{c\pi h} \times \frac{\sin \frac{n\pi}{2}}{n^2 \pi^2}$$

Here, eqn (1) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

$$= \sum_{n=1}^{\infty} \left[\frac{1600 \sin \frac{n\pi}{2}}{c^2 \pi^3 h^3} \cdot \sin \frac{c n \pi t}{20} \right] \sin \frac{n\pi}{20} x$$



Exam:

Q. Solve one dim. wave eqn with initial deflection is $0.01 \sin 3x$ and initial velocity is zero at $L=\pi$, $c^2=L$.

$\Rightarrow$ We have dim. wave eqn is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Given, then the soln this eqn according with question,

$u(x,0) = f(x) = 0.01 \sin 3x$, $g(x) = 0$ $L=\pi$ $c^2=L$
then,

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \sin \lambda_n t) \sin \frac{n\pi}{L} x \quad [\because B_n^* = 0]$$

Also,

$$u(x,0) = \sum_{n=1}^{\infty} (B_n \sin \frac{n\pi}{L} x)$$

$$= \sum_{n=1}^{\infty} (B_n \sin nx) \quad [\because L=\pi]$$

$$0.01 \sin 3x = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + B_4 \sin 4x + \dots$$

Comparing all,

$$B_3 = 0.01, B_1 = B_2 = B_4 = \dots = 0$$

Hence the required solⁿ is,

$$U(x, t) = (B_3 \cos 3t) \sin 3x$$

$$= 0.01 \cos 3t \sin 3x \quad [\because t = 18]$$

Solve the differential eqⁿ :-
 $U_x + U_y = 0$ using Separating variable.

⇒ let the solⁿ of the given differential equation is
 $U(x, y) = f(x) \cdot g(y)$ — (i)
 Diff. ~~partial~~ partial 'u' w.r. to x and y respectively.

$$U_x = \frac{\partial U}{\partial x} = f'(x)g(y)$$

$$U_y = \frac{\partial U}{\partial y} = f(x)g'(y)$$

Where dash denote the derivative.

Then the original eqⁿ becomes,

$$f'(x)g(y) + f(x)g'(y) = 0 \quad [\because U_x + U_y = 0]$$

$$f'(x)g(y) = -f(x)g'(y)$$

$$\frac{f'(x)}{f(x)} = -\frac{g'(y)}{g(y)} = k \text{ (say)}$$

Now taking first and last ratio.

$$\frac{f'(x)}{f(x)} = k$$

taking integration on both sides,

$$\int \frac{f'(x)}{f(x)} = \int k dx$$

$$\Rightarrow \ln f = kx + c$$

$$f = e^{kx+c}$$

$$f = e^{kx} \cdot e^c$$

$$= Ae^{kx} \quad \text{--- (ii) Where } A = e^c$$

Again,

taking second and last term,

$$\frac{g'(y)}{g(y)} = -k$$

taking integration.

$$\ln g = -ky + D$$

$$g = e^{-ky+D}$$

$$= e^{-ky} \cdot e^D$$

$$= B e^{-ky} \quad \text{where } B = e^D$$

Hence,

$$U(x, y) = G \cdot F = A e^{kx} \cdot B e^{-ky}$$

$$= C e^{kx - ky} \quad \#$$

where $C = AB$.

hw

(a) $y^2 v_x - x^2 v_y = 0$ (b) $U_{xx} + U_{yy} = 0$

(b) $U_{xx} + U_{yy} = 0$

$\Rightarrow$ let the solⁿ of the given eqⁿ is
 $U(x, y) = F(x) \cdot G(y) \quad \text{--- (i)}$

Diff. parti. 'U' w.r. to x & y respectively
 then,

$$U_x = \frac{\partial U}{\partial x} = F'G \quad \text{--- (a)}$$

$$U_y = \frac{\partial U}{\partial y} = FG' \quad \text{--- (b)}$$

Again, Diff. p. eqⁿ (a) & (b) w.r. to x & y.

$$U_{xx} = F''G$$

$$U_{yy} = FG''$$

where dash denotes derivatives,

Now, eqⁿ (i) becomes,

$$F''G + FG'' = 0$$

$$F''G = -FG''$$

$$\frac{F''}{F} = -\frac{G''}{G} = k \text{ (say)}$$

Taking first and last term, and taking integⁿ on both side.

~~$$\frac{F''}{F} = -\frac{G''}{G} = k$$~~

$$\frac{F''}{F} = k$$

$$\Rightarrow F'' = kF$$

$$\Rightarrow F'' - kF = 0$$

$$m^2 - k = 0$$

$$m = \pm \sqrt{k} \quad \text{then the eqⁿ becomes,}$$

$$F = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x} \quad \text{--- (2)}$$

Again, taking second and last term,

initial deflection = $f(x) = U(x,0) = 0$
 initial velocity = $g(x) = B_n^*$

$$\frac{G''}{G} = -k$$

$$\Rightarrow G'' + kG = 0$$

$$m^2 + k = 0$$

$$m^2 = -k$$

$$G = A \cos \sqrt{k} y + B \sin \sqrt{k} y \quad (*)$$

Now, the required soln is,

$$U(x,y) = (C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}) \cdot (A \cos \sqrt{k} y + B \sin \sqrt{k} y)$$

#

Ex. 5) Find the deflection $U(x,t)$ of the vibrating string of length $L = \pi$, $c^2 = 1$ and it's initial deflection is zero and initial velocity is,

$$\begin{cases} 0.01x & \text{if } 0 < x < \pi/2 \\ 0.01(\pi-x) & \text{if } \pi/2 < x < \pi \end{cases}$$

$\Rightarrow$ let the required soln of the given diff. eqn.

G.P. 2, 3, 4, 5, 7.

$$U(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

Since, initial deflection $f(x) = 0 \Rightarrow B_n = 0$
 and initial velocity $g(x) = \begin{cases} 0.01x & \text{if } 0 < x < \pi/2 \\ 0.01(\pi-x) & \text{if } \pi/2 < x < \pi \end{cases}$

Hence, the eqn soln/deflection $U(x,t)$ of string is,

$$U(x,t) = \sum_{n=1}^{\infty} B_n^* \sin \lambda_n t \sin \frac{n\pi}{L} x \quad (*)$$

$$B_n^* = \frac{2}{L \lambda_n} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{L \lambda_n} \left[\int_0^{\pi/2} 0.01x \sin nx \, dx + \int_{\pi/2}^{\pi} 0.01(\pi-x) \sin nx \, dx \right]$$

$$= \frac{2}{L \lambda_n} \left[\left[0.01x \frac{-\cos nx}{n} - 0.01 \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi/2} + \left[0.01\pi \frac{-\cos nx}{n} - 0.01x \left(\frac{-\cos nx}{n} \right) + 0.01 \frac{\sin nx}{n^2} \right]_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{L \lambda_n} \left[0.01 \right]$$

for me.

$\triangleright$ 'Alembert's solution of wave eqn']

H/w] one dim. heat eqn and its solution.

$$\Rightarrow \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{k}{\rho s} \quad (\text{diffusivity of material})$$

It's soln.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x e^{-\alpha_n^2 t}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx.$$

Determine the soln of one dim. heat eqn under the boundary condⁿ $u(0,t) = 0$, $u(L,t) = 0$ & initial temp^r is $u(x,0) = x$, where L is the length of bar.

$\Rightarrow$ By the condⁿ of one dim. heat eqn,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x e^{-\alpha_n^2 t}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx.$$

Where $f(x) = x$.

then,

$$B_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi}{L} x dx$$

Q.N. 3, 2, 4, 5

$$= \frac{2}{L} \left[a \left(-\cos \frac{n\pi}{L} x \right) - \frac{1 \cdot \left(-\sin \frac{n\pi}{L} x \right)}{\left(\frac{n\pi}{L} \right)^2} \right]_0^L$$

$$= \frac{2}{L} \left[-L \frac{\cos n\pi}{n\pi} + 0 \right]$$

$$= -\frac{2L}{n\pi} \cos n\pi$$

Hence,

$$V(x, t) = \sum_{n=1}^{\infty} \left(-\frac{2L}{n\pi} \right) \cos n\pi \sin \frac{n\pi}{L} x e^{-\lambda_n t}$$

Where $\lambda_n = \frac{cn\pi}{L}$

#.

H.W Q.N. 2, 3, 4, 5, 7

2. Find the deflection $v(x, t)$ of a vibrating string of length π and $c^2 = 4$ for zero initial velocity and initial deflection $\sin 5x$.

⇒ Given,

length $(L) = \pi$

$c^2 = 4 \Rightarrow c = \pm 2$

initial velocity $g(x) = 0$

initial deflection $f(x) = v(x, 0) = \sin 5x$

We have,

one dim. wave eqⁿ is

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} \quad \text{then the}$$

solution of eqⁿ is.

$$V(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t) \sin \frac{n\pi}{L} x \quad \text{--- (1)}$$

Also, $V(x, 0) = \sum_{n=1}^{\infty} (B_n \sin n\pi x) \quad [\because t=0, L=\pi]$

$$\sin 5x = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + B_4 \sin 4x + B_5 \sin 5x + \dots$$

Comparing all,

$$B_5 = 1, B_1 = B_2 = B_3 = B_4 = B_6 = \dots = 0$$

hence the required solⁿ is,

$$v(x, t) = (B_5 \cos \lambda_5 t) \sin 5x$$

$$= -\cos 10t \sin 5x \quad \# \quad [\because \lambda_n = \frac{cn\pi}{L} = \frac{2 \cdot 5 \cdot \pi}{\pi} = 10]$$

Exam

find the z-transform of $e^{in\pi/2}$ and then $z(\cos \frac{n\pi}{2})$ and $z(\sin \frac{n\pi}{2})$

$\Rightarrow$ We know that,

$$z(f(n)) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\Rightarrow z(e^{in\pi/2}) = \left(\sum_{n=0}^{\infty} e^{in\pi/2} z^{-n} \right) * \\ = z(e^{i\pi/2})^n$$

$$= \frac{z}{z - e^{i\pi/2}} \left[\because z[a]^n = \frac{z}{z-a} \right]$$

$$\Rightarrow z\left(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}\right) = \frac{z}{z - (\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})}$$

Theorem: $z(f^k) = -T_z \frac{d(z + k \cdot 1)}{dz}$

Put $k=1$

$$\Rightarrow z(f) = -T_z \frac{d(z + 1)}{dz}$$

$$= -T_z d\left(\frac{z}{z-1}\right)$$

$$= -T_z \times \frac{dz}{(z-1) \cdot 1 - z \cdot 1}$$

$$= \frac{T_z}{(z-1)^2}$$

Put $k=2$

$$z(f^2) = -T_z \frac{d}{dz} z(f)$$

$$= -T_z \frac{d}{dz} \frac{T_z}{(z-1)^2}$$

$$= T_z^2 \left[\frac{z+1}{(z-1)^3} \right]$$

State and prove the first shifting theorem of Z-transform.

⇒ Statement: If $Z[f(t)] = f(z)$ then
 $Z[e^{-at} f(t)] = F[ze^{at} z]$

Proof: We have by definition,

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad \text{--- (i)}$$

$$\begin{aligned} \Rightarrow Z[e^{-at} f(t)] &= \sum_{n=0}^{\infty} f(nT) (ze^{at})^{-n} \\ &= \sum_{n=0}^{\infty} [ze^{at}]^n f(nT) z^{-n} \quad \text{or.} \\ &= \sum_{n=0}^{\infty} f(nT) [ze^{at} z]^{-n} \\ &= Z[ze^{at} z] \quad \text{by definition from (i)} \end{aligned}$$

Exam.

Q. Find the Z-transform of $z(e^{iat})$ and find the relation $Z(\cos at)$ & $Z(\sin at)$

⇒ We have,
 $Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$\text{and, } Z(e^{iat}) = Z(e^{iat} \cdot 1)$$

$$= Z(1)_{z \rightarrow ze^{iat}}$$

$$= \left[\frac{z}{z-1} \right]_{z \rightarrow ze^{iat}}$$

$$= \frac{ze^{iat}}{ze^{iat} - 1}$$

$$= \frac{e^{iat} z}{e^{iat} (z - e^{-iat})}$$

$$= \frac{z}{z - (\cos at - j \sin at)}$$

$$= \frac{z}{(z - \cos at) + j \sin at}$$

$$\frac{z(z - \cos a) - i \sin a}{(z - \cos a)^2 + \sin^2 a}$$

$$= \frac{z^2 - z \cos a - i \sin a}{(z - \cos a)^2 + \sin^2 a}$$

$$\Rightarrow \mathcal{Z}[\cos a] + i \mathcal{Z}[\sin a] = \frac{z^2 - z \cos a - i \sin a}{z^2 - 2z \cos a + \cos^2 a + \sin^2 a}$$

$$= \frac{z^2 - z \cos a - i \sin a}{z^2 - 2z \cos a + 1}$$

$$= \frac{z^2 - z \cos a}{z^2 - 2z \cos a + 1} - i \frac{\sin a}{z^2 - 2z \cos a + 1}$$

Hence,

$$\mathcal{Z}(\cos a) = \frac{z - \cos a}{z^2 - 2z \cos a + 1}$$

$$\mathcal{Z}(\sin a) = \frac{z \sin a}{z^2 - 2z \cos a + 1} \quad \#$$

Example.

Q. Find $\mathcal{Z}(\cosh a \cdot \sin b)$

$$\Rightarrow \mathcal{Z}\left[\frac{e^{at} + e^{-at}}{2} \sin bt\right]$$

$$= \frac{1}{2} \left\{ \mathcal{Z}[e^{at} \sin bt] + \mathcal{Z}[e^{-at} \sin bt] \right\}$$

$$= \frac{1}{2} \left\{ \mathcal{Z}(\sin bt)_{z \rightarrow ze^{-at}} + \mathcal{Z}(\sin bt)_{z \rightarrow ze^{at}} \right\}$$

$$= \frac{1}{2} \left\{ \left[\frac{z \sin b}{z^2 - 2z \cos b + 1} \right]_{z \rightarrow ze^{-at}} + \left[\frac{z \sin b}{z^2 - 2z \cos b + 1} \right]_{z \rightarrow ze^{at}} \right\}$$

$$= \frac{1}{2} \left\{ \frac{e^{-at} \sin b}{e^{-2at} - 2e^{-at} \cos b + 1} + \frac{e^{at} \sin b}{e^{2at} - 2e^{at} \cos b + 1} \right\}$$

#

Imp Theorem:
 $\mathcal{Z}[a^n f(n)] = f\left(\frac{z}{a}\right)$ for $\mathcal{Z}[f(n)] = F(z)$

Example:
 find the z -transform of $\mathcal{Z}(na^n)$
 We know that, $\mathcal{Z}[a^n f(n)] = f\left(\frac{z}{a}\right)$

$$\begin{aligned} \Rightarrow \mathcal{Z}[a^n \cdot n] &= \mathcal{Z}[n]_{z \rightarrow (z/a)} \\ &= \left[\frac{z}{(z-1)^2} \right]_{z \rightarrow (z/a)} \\ &= \frac{z/a}{(z/a - 1)^2} \\ &= \frac{az}{(z-a)^2} \quad \# \end{aligned}$$

Example:
 find the z -transform of $\mathcal{Z}(n-1)$
 $= \mathcal{Z}(n) - \mathcal{Z}(1)$

$$\begin{aligned} &= \frac{z}{(z-1)^2} - \frac{z}{(z-1)} \\ &= \frac{z - z(z-1)}{(z-1)^2} \end{aligned}$$

$$\begin{aligned} &= \frac{z - z^2 + z}{(z-1)^2} \\ &= \frac{z - z^2}{(z-1)^2} \quad \# \end{aligned}$$

State and prove the second shifting theorem.
 $\Rightarrow$ Statement: If $\mathcal{Z}[f(n)] = F(z)$ then $\mathcal{Z}[f(n+1)] = \mathcal{Z}[f(z) - f(0)]$

Proof: We know by defn;

$$\mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\begin{aligned} \therefore \mathcal{Z}[f(n+1)] &= \sum_{n=0}^{\infty} f(n+1) z^{-n} \\ &= \sum_{n=0}^{\infty} f[(n+1)] z^{-n} \end{aligned}$$

Put $N = n+1$ then,

$$\begin{aligned} \mathcal{Z}[f(n+1)] &= \sum_{N=1}^{\infty} f(N) z^{-(N-1)} - f(0) = \mathcal{Z}[f(z) - f(0)] \quad \text{proved.} \end{aligned}$$

Convolution theorem: (self)

Inverse z-transform: by partial fraction.

① if $f(z) = \frac{h(z)}{(az+b)(z+d)} = \frac{Az}{(az+b)} + \frac{Bz}{(z+d)}$

② if $f(z) = \frac{h(z)}{(az+b)(z^2+d)} = \frac{Az}{(az+b)} + \frac{Bz^2+Cz}{(cz^2+d)}$

③ if $f(z) = \frac{h(z)}{(az+b)(cz+d)^2} = \frac{Az}{(az+b)} + \frac{Bz}{(cz+d)} + \frac{Cz}{(cz+d)^2}$

find $z^{-1} \left(\frac{z}{z^2-5z+6} \right)$

$\Rightarrow \frac{z}{(z^2-5z+6)} = \frac{z}{(z-2)(z-3)}$

$\Rightarrow \frac{z}{(z-2)(z-3)} = \frac{Az}{(z-2)} + \frac{Bz}{(z-3)}$

$\Rightarrow z = Az(z-3) + Bz(z-2)$

$\therefore A = -1, B = 1$

i.e. $\mathcal{Z}(a^n) = \frac{z}{z-a}$

Now, $z^{-1} \left(\frac{z}{z^2-5z+6} \right) = z^{-1} \left(\frac{-z}{(z-2)} \right) + z^{-1} \left(\frac{z}{(z-3)} \right)$
 $= (-1) z^{-1} \left(\frac{z}{z-2} \right) + z^{-1} \left(\frac{z}{z-3} \right)$
 $= (-1) 2^n + 3^n$
 $= 3^n - 2^n$ #

Note: To find inverse of z-transform to use residues method as.

$z^{-1} [f(z)] = f[n] = \text{sum of residues of } f(z) \cdot z^{n-1}$

Find the inverse z-transform by residue method: $\frac{z}{z^2 + 7z + 10}$

⇒ We have.

$$f(nT) = \frac{1}{2\pi j} \int_C z^{n-1} f(z) dz$$

$$= \frac{1}{2\pi j} \int_C \frac{z^{n-1} \cdot z}{(z+5)(z+2)} dz$$

$$= \frac{1}{2\pi j} \int_C \frac{z^n}{(z+5)(z+2)} dz$$

The given poles: $z^2 + 7z + 10 = 0 \Rightarrow z = -5, -2$
Which are simple poles

$$\begin{aligned} \text{Res. of } f(z) \text{ at } z = -2 &\Rightarrow \text{Res}[f(z), -2] = \lim_{z \rightarrow -2} [(z+2) f(z)] \\ &= \lim_{z \rightarrow -2} \left[\frac{z^n}{(z+5)} \right] \\ &= \frac{(-2)^n}{3} \end{aligned}$$

Again.

$$\text{Res. of } f(z) \text{ at } z = -5 \Rightarrow \text{Res}[f(z), -5] = \lim_{z \rightarrow -5} [(z+5) f(z)]$$

① formula
② shifting first
③ second shifting
④ Inverse
⑤ marks.
 $= \frac{(-5)^n}{-3}$

Now for, $f(nT) = \text{sum of all residues.}$

$$= \frac{(-2)^n}{3} + \frac{(-5)^n}{-3}$$

$$= \frac{(-2)^n - (-5)^n}{3} \quad \#$$

Difference equation:

① Define ordinary difference eqn: The eqn which contains one independent variable x and one or more dependent variables with any successive differences is called ordinary difference eqn.

Example:

$$5y_{x+2} + 3y_{x+1} = 7$$

$$\text{or. } y_{x+2} - 3y_{x+1} - 4y_x = 6$$

④ The degree of difference eqⁿ.

$y_{x+1} - y_x = 5$ is of degree one

$y_{x+2}^3 + 2y_{x+1} - y_x = 0$ is of degree third

Application of difference equation.

Theorem: $z(y_{n+k}) = z^k \left[\bar{y} - y_0 - \frac{y_1}{z} - \dots - \frac{y_{k-1}}{z^{k-1}} \right]$

where

$$\bar{y} = z(y_k)$$

We need:

$$z(y_{k+1}) = z(\bar{y} - y_0)$$

$$z(y_{k+2}) = z^2 \left(\bar{y} - y_0 - \frac{y_1}{z} \right)$$

$$z(y_{k+3}) = z^3 \left(\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2} \right)$$

Solve the difference equation:-

$$y_{k+1} + y_k = 1 \quad y_0 = 0$$

$$\Rightarrow y_{k+1} + y_k = 1, \quad y_0 = 0$$

Taking z-transform on both sides.

$$z(y_{k+1}) + z(y_k) = z(1)$$

$$z(\bar{y} - y_0) + \bar{y} = \frac{z}{z-1} \quad \left[\because z(1) = \frac{z}{z-1} \right]$$

$$z\bar{y} - 0 + \bar{y} = \frac{z}{z-1}$$

$$\bar{y}(z+1) = \frac{z}{(z-1)}$$

$$\bar{y} = \frac{z}{(z-1)(z+1)} = \frac{Az}{(z-1)} + \frac{Bz}{(z+1)}$$

$$z = Az(z+1) + Bz(z-1)$$

$$\therefore A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\bar{y} = \left(\frac{1}{2} \right) \frac{z}{(z-1)} - \left(\frac{1}{2} \right) \frac{z}{(z+1)}$$

$$\Rightarrow Z(Y_k) = \left(-\frac{1}{2}\right) \left(\frac{Z}{Z+1}\right) + \left(\frac{1}{2}\right) \left(\frac{Z}{Z-1}\right)$$

$$Y_k = -\frac{1}{2} Z^{-1} \left(\frac{Z}{Z+1}\right) + \frac{1}{2} Z^{-1} \left(\frac{Z}{Z-1}\right)$$

$$Y_k = \left(-\frac{1}{2}\right) (-1)^k + \left(\frac{1}{2}\right) (1)^k$$

$$\therefore Y_k = \left(-\frac{1}{2}\right) (-1)^k + \frac{1}{2}$$

Q. $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ and $y_0 = y_1 = 0$

$\Rightarrow$ Given

$$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$$

Taking Z-transform on both side,

$$Z(Y_{n+2}) + 6Z(Y_{n+1}) + 9Z(Y_n) = Z(2^n)$$

$$\Rightarrow Z^2(\bar{y} - y_0 - \frac{y_1}{Z}) + 6[Z(\bar{y} - y_0)] + 9\bar{y} = \frac{Z}{Z-2}$$

$$\Rightarrow Z^2(\bar{y}) + 6Z(\bar{y}) + 9\bar{y} = \frac{Z}{Z-2}$$

$$\Rightarrow \bar{y} (Z^2 + 6Z + 9) = \frac{Z}{Z-2}$$

$$\Rightarrow \bar{y} = \frac{Z}{(Z-2)(Z+3)^2} \quad (4)$$

The poles of $\frac{Z}{(Z-2)(Z+3)^2} = 0$
 $\Rightarrow Z = 2, -3$

Res. at $Z=2$ at a simple pole. then.

$$\text{Res}[f(Z), 2] = \lim_{Z \rightarrow 2} \left[(Z-2) \frac{Z}{(Z-2)(Z+3)^2} \right]$$

$$= \frac{2}{25}$$

Again,
 Res. at $Z=-3$ at simple pole of order 2 then

$$\text{Res}[f(Z), -3] = \lim_{Z \rightarrow -3} \frac{d}{dZ} \left[(Z+3)^2 \frac{Z}{(Z-2)(Z+3)^2} \right]$$

$$= \lim_{Z \rightarrow -3} \frac{d}{dZ} [Z(Z-2)]$$

$$= \lim_{Z \rightarrow -3} [Z(-1)(Z-2) + (Z-2) \cdot 1]$$

$$2 \lim_{z \rightarrow -3} \left[\frac{-2}{(z-2)^2} + \frac{1}{(z+2)} \right]$$

$$= \frac{3}{25} + \frac{1}{(-5)} = \frac{3}{25} - \frac{1}{5}$$

$$= \frac{3-5}{25} = \frac{-2}{25}$$

$$u_{xx} = (u_r r_x) + (u_\theta \theta_x)$$

$$= (u_r) r_x + u_r r_{xx} + (u_\theta) \theta_x + u_\theta \theta_{xx}$$

$$= (u_{rr} r_x + u_{r\theta} \theta_x) r_x + u_r r_{xx} +$$

$$(u_{\theta r} r_x + u_{\theta\theta} \theta_x) \theta_x + u_\theta \theta_{xx}$$

$$r_{xx} = \frac{r - x r_x}{r^2} = \frac{1}{r} - \frac{2}{r^3} = \frac{y^2}{r^3}$$

$$\theta_{xx} = -y \left(-\frac{2}{r^3} \right) r_x = \frac{2xy}{r^4}$$

$$u_{xx} = \frac{2}{r^2} u_{rr} - \frac{2xy}{r^3} u_{r\theta} + \frac{1}{r^4} u_{\theta\theta} + \frac{y^2}{r^3} u_r + \frac{2xy}{r^4} u_\theta - \text{(iii)}$$

similarly,

$$u_{yy} = \frac{1}{r^2} u_{rr} + \frac{2xy}{r^3} u_{r\theta} + \frac{x^2}{r^4} u_{\theta\theta} + \frac{x^2}{r^3} u_r$$

$$- \frac{2xy}{r^4} u_\theta - \text{(iv)}$$

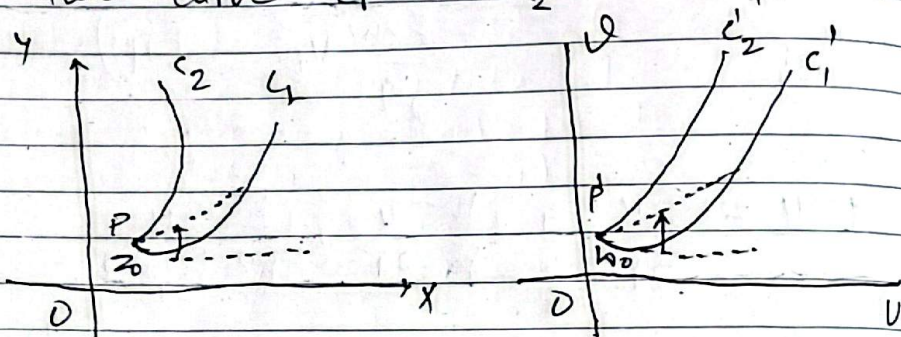
f.

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

#

Linear Transformation [8 marks]

- ① Conformal mapping: Consider the transformation $w = f(z)$. Where $f(z)$ is single valued function of z . Under the transformation, a point z_0 and any two curves C_1 and C_2 passing through z_0 in the z -plane will be mapped into a point w_0 and two curves C'_1 and C'_2 in w plane.



Def: Define Conformal mapping: A transform that preserves angles between every pair of curves through a point, both in magnitude and sense, is said to be Conformal at that point.

- ② Isogonal mapping: A transformation under which angles between the every pair of curves through a point both in magnitude and sense are preserved in magnitude but not in sense is called isogonal mapping.

Note: 1. A mapping $w = f(z)$ is said to be Conformal at $z = z_0$ if $f'(z_0) \neq 0$

2. The point at which mapping $w = f(z)$ is not conformal if $f'(z) = 0$ is called critical point of the mapping.

Some Standard transformation:-

1. Translation
2. Magnification
3. Magnification and Rotation.
4. Magnification, Rotation and Translation
5. Inversion and Reflection.

1. Translation: The transformation $w = c + z$, Where c is a complex number then,

$$z = x + iy, w = u + iv, c = a + ib$$

$$\Rightarrow u + iv = a + ib + x + iy$$

$$\Rightarrow u + iv = (a+x) + i(b+y)$$

Equating,

$$u = (a+x)$$

$$v = (b+y)$$

Ex. What is the region of w plane where the z plane is bounded by the line $x=0$, $y=0$, $x=1$, $y=2$. Under the mapping $w = z(2-i)$

$$\Rightarrow x=0, y=0, x=1, y=2$$

$$\text{So } w = z + (2-i)$$

$$u + iv = x + iy + (2-i)$$

$$= (x+2) + i(y-1)$$

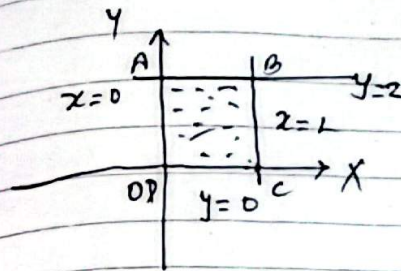
$$\Rightarrow u = x+2$$

$$v = y-1$$

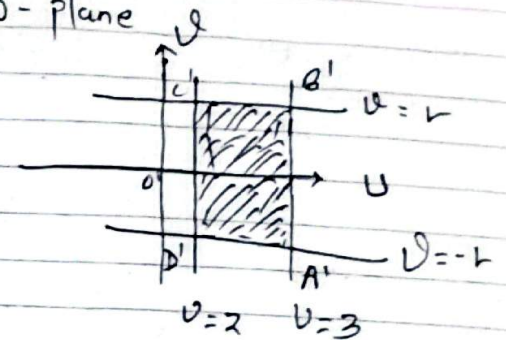
$$\text{When, } x=0 \Rightarrow u=2, x=1 \Rightarrow u=3$$

$$y=0 \Rightarrow v=-1, y=2 \Rightarrow v=1$$

z -plane



w -plane



2.] Magnification: The transformation $w = Cz$, where C is real constant, represent magnification

$$u + iv = c(x + iy) = cx + icy$$

$$u = cx, v = cy$$

3.] Magnification and rotation: The transformation $w = Cz$ where C is complex constant, represent both magnification and rotation,

$$z = re^{i\theta}$$

$$w = Re^{i\phi}$$

$$C = ce^{i\alpha}$$

then, $w = cz$
 $Re^{i\psi} = ae^{i\alpha} \cdot re^{i\theta}$

$Re^{i\psi} = ar(e^{i(\alpha+\theta)})$
 Now, equating,

$R = ar, \quad \psi = \alpha + \theta$

$\Rightarrow (r, \theta)$ is mapped on the point $(ar, \alpha + \theta)$

5] Inversion and Reflection: The transformation $w = \frac{1}{z}$, represent inversion with 1 to

the unit circle $|z| = 1$ followed by reflection in real axis,

$w = \frac{1}{z}$ or, $z = \frac{1}{w}$

$\Rightarrow x + iy = \frac{1}{u + iv}$

$= \frac{u - iv}{u^2 + v^2} = \frac{u}{u^2 + v^2} - i \frac{v}{u^2 + v^2}$

Equating, $x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}$

Ex. find the image of $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$.

$\Rightarrow$ Solⁿ: ~~$|z - 2i| = 2$~~

$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

then, $\Rightarrow z = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}$

Again,

$|z - 2i| = 2$

$|x + iy - 2i| = 2$

$|x + i(y - 2)| = 2$

$\sqrt{x^2 + (y - 2)^2} = 2$

$x^2 + (y - 2)^2 = 4$ — (i)

$x^2 + y^2 - 4y + 4 = 4$

$x^2 + y^2 - 4y = 0$

$$\frac{v^2}{(v^2+v^2)^2} + \frac{v^2}{(v^2+v^2)^2} + \frac{4v}{v^2+v^2} = 0$$

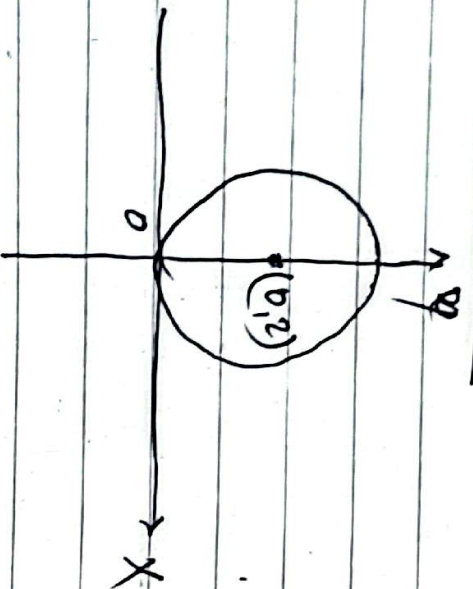
$$\text{or, } \frac{v^2+v^2+4v(v^2+v^2)}{(v^2+v^2)^2} = 0$$

$$\text{or, } \frac{v^2+v^2+4v(v^2+v^2)}{(v^2+v^2)^2} = 0$$

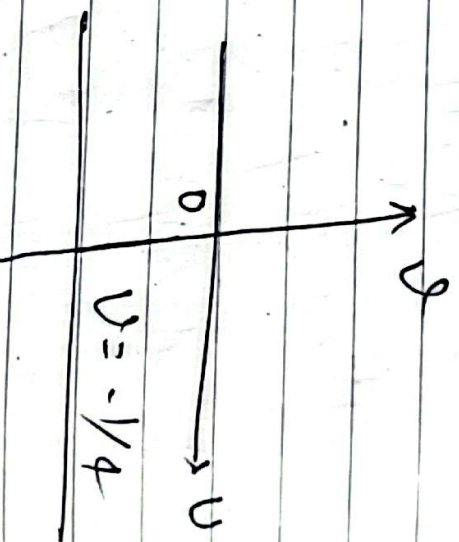
$$\text{or } 1+4v=0$$

$$\Rightarrow v = -1/4$$

z-plane



w-plane



Practice:

Exercise - 1.1

1. (d) $z_1 \bar{z}_2$ and $\bar{z}_1 z_2$

$\Rightarrow$ We have,

$$z_1 = 4 + 3i, z_2 = 2 - 5i$$

here, $z_1 \bar{z}_2$

$$\Rightarrow (4 + 3i) \cdot (2 + 5i)$$

$$= (4 + 3i)(2 + 5i)$$

$$= 8 + 20i + 6i - 15$$

$$= -7 + 26i$$

Comparing, $f(z) = x + iy$

where

$$x = -7$$

$$y = 26$$

$$\} \neq$$

$$2. (c) \frac{(4 + 5i)^2}{(2 + 3i)^2} \Rightarrow \frac{16 + 2 \cdot 4 \cdot 5i - 25}{4 + 2 \cdot 2 \cdot 3i - 9}$$

$$= \frac{(-9 + 40i)}{(-5 + 12i)} \times \frac{(-5 - 12i)}{(-5 - 12i)}$$

$$= \frac{45 + 108i - 200i + 480}{25 + 144}$$

$$= \frac{525 - 92i}{169}$$

$$= \frac{525}{169} - \frac{92i}{169} \neq$$

$$(r \cos \theta + i \sin \theta)$$

3

formula:

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

or,

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

$$3(f) \quad 1 + \sin \alpha + i \cos \alpha$$

⇒ Here.

Given, $1 + \sin \alpha + i \cos \alpha$

We know.

$$f(z) = x + iy \quad \text{Equating,}$$

$$x = 1 + \sin \alpha \quad y = \cos \alpha.$$

We know, $r = \sqrt{x^2 + y^2}$

$$= \sqrt{(1 + \sin \alpha)^2 + \cos^2 \alpha}$$

$$= \sqrt{1^2 + 2\sin \alpha + \sin^2 \alpha + \cos^2 \alpha}$$

$$= \sqrt{2 + 2\sin \alpha}$$

$$= \sqrt{2(1 + \sin \alpha)}$$

$$= \sqrt{2 \cdot 2 \cos^2 \frac{\alpha}{2}}$$

$$= 2 \cos \frac{\alpha}{2}$$

$$\theta = \tan^{-1} \left(\frac{\cos \alpha}{1 + \sin \alpha} \right)$$

4.(d) Given.

$$\sqrt{18} \left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right]$$

We know, $f(z) = r[\cos \theta + i \sin \theta]$

Where, $r = \sqrt{18}$

$$\theta = \frac{3\pi}{4}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{18}$$

S.q. on both side.

$$x^2 + y^2 = 18 \quad \text{--- (i)}$$

$$\cancel{\theta = \tan^{-1} \left(\frac{y}{x} \right)} \quad \tan \theta = \left(\frac{y}{x} \right)$$

$$\tan \left(\frac{3\pi}{4} \right) = \left(\frac{y}{x} \right)$$

$$-1 = \frac{y}{x}$$

$$-x = y \quad \text{--- (ii)}$$

$$x^2 + (-x)^2 = 18$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = 3, \quad y = -3$$

Now, $f(z) = x + iy = 3 - 3i$

(5) (c) $-11 - 11i$
 $\Rightarrow f(z) = x + iy$
 Equating $x = -11$
 $y = -11$

$$r = \sqrt{(-11)^2 + (-11)^2} = 16\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-11}{-11}\right) =$$

Formula: $1 + \tan^2 \alpha = \sec^2 \alpha$ (4)
 $\theta = \tan^{-1}(\tan \alpha) = \alpha$

(2) $1 + \cos \alpha = 2 \cos \frac{\alpha}{2}$

$$\theta = \tan^{-1}\left(\frac{\tan \alpha}{1 + \cos \alpha}\right) = \tan^{-1}\left(\frac{\tan \alpha}{2 \cos \frac{\alpha}{2}}\right)$$

$$= \frac{\alpha}{2}$$

3. (1) $\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha} \Rightarrow \frac{\cos \alpha}{1 + \sin \alpha}$
 $= \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

$$\theta = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right)$$

$$\theta = \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$

$$\downarrow r \cos \theta + i \sin \theta = \left(\frac{1}{2} \right)$$

$\frac{\pi}{2}$

(5.c) Where both real and imaginary part are negative. This means the complex number lies ~~between~~ in third quadrant. In third quadrant, we add π to the principle value.

principle value (restricted $(-\pi, \pi]$)

$$= -\frac{3\pi}{4}$$

Because

$$\frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4} \quad \left(\text{to bring into } (-\pi, \pi]\right)$$

Formula:

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$-2 \sin A \sin B = \cos(A+B) - \cos(A-B)$$

$$\text{or, } \sin(A+B) + \sin(B-A)$$

$$2 \cos A \sin B =$$

1. Cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(\omega) \cos \omega x d\omega; \quad f_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t dt.$$

(1) Complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{for } (-\pi < x < \pi)$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

(2) Fourier integral

$$f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + \sin \omega x \cdot B(\omega)] d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t dt;$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t dt.$$

(3) Fourier the sine and cosine integral:-
Sine integral

$$f(x) = \int_0^{\infty} B(\omega) \sin \omega x d\omega, \quad B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \sin \omega t dt.$$

Cosine integral

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x d\omega, \quad A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos \omega t dt$$

(4) Fourier sine and cosine transform.

Fourier sine transform.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(\omega) \sin \omega x d\omega; \quad f_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t dt$$