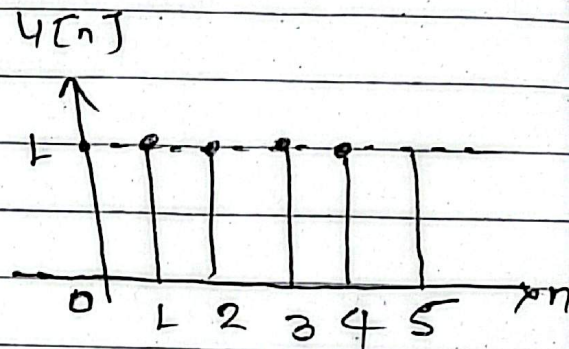


#. Elementary signals (Basic signals)

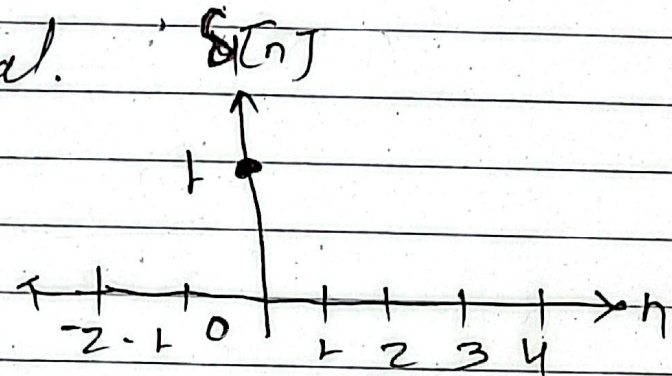
- 1) Unit step signal ✓
- 2) unit impulse (Delta) signal
- 3) Ramp signal ✓
- 4) Exponential signal
- 5) Sinusoidal signal
- 6) Signum signal ✓

- 1) Unit step signal.

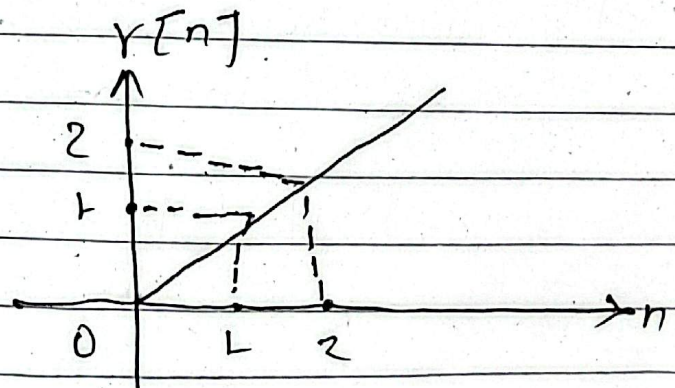
$$u[n] = \begin{cases} 1; & \text{for } n \geq 0 \\ 0; & \text{otherwise} \end{cases}$$

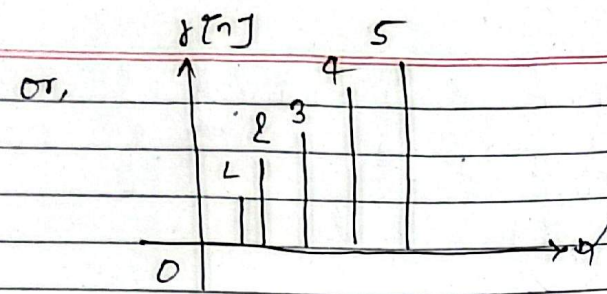


- 2) unit impulse (Delta) signal.
It is defined as.
 $\delta[n] = \begin{cases} 1; & \text{for } n = 0 \\ 0; & \text{otherwise} \end{cases}$



- 3) Ramp signal.
It is defined as
 $r[n] = \begin{cases} n; & \text{for } n \geq 0 \\ 0; & \text{otherwise} \end{cases}$





Graphical representation of Ramp signal

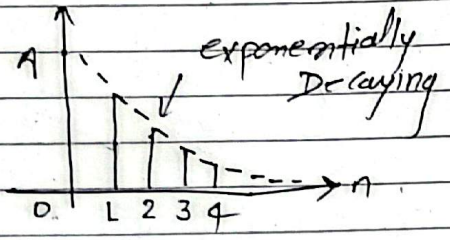
4) Exponential Signals.

It is defined as,

$$x[n] = Ae^{-\alpha n} ; \text{for } n \geq 0 \text{ (Decaying signal)}$$

or

$$x[n] = Ae^{\alpha n} ; \text{for } n \geq 0 \text{ (growing signal)}$$



5) Sinusoidal Signal :-

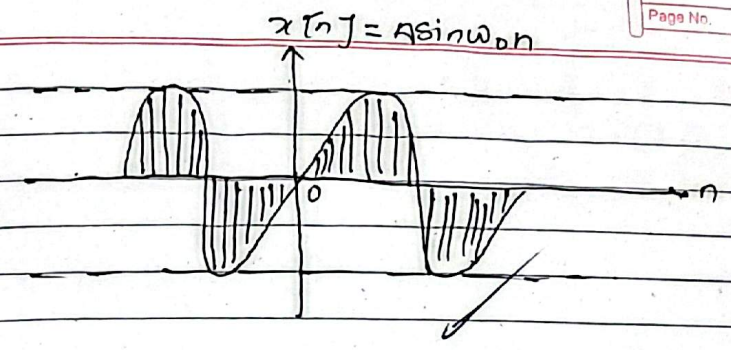
↳ It includes trigonometric funcⁿ Sine and Cosine

↳ it is defined as,

$$x[n] = A \sin \omega_0 n, \quad -\infty < n < \infty$$

or

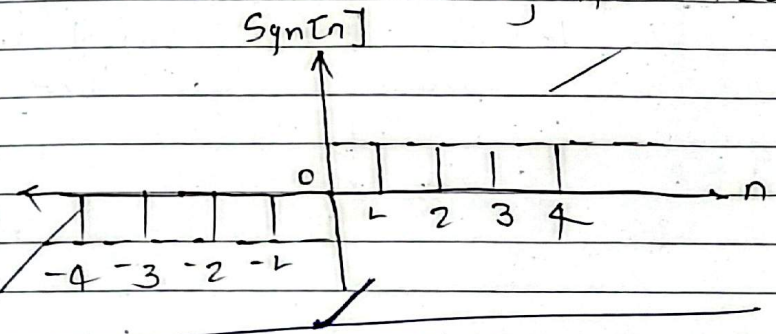
$$A \cos \omega_0 n, \quad -\infty < n < \infty$$



b) Signum signal:

It is defined as,

$$x[n] = \text{sgn}[n] = \begin{cases} 1 & \text{for } n > 0 \\ 0 & \text{for } n = 0 \\ -1 & \text{for } n < 0 \end{cases}$$



Additional signal :-

a) Even / odd signal

b) periodic / non-periodic (Aperiodic)

c) Energy and Power

d) Deterministic / non-Deterministic (Random)

a) Even/Odd Signal
→ Even Signal,

$$x[n] = x[-n]$$

→ Odd Signal,

$$x[n] = -x[-n]$$

b) Periodic and Non-periodic. Difference.

Periodic

→ They repeat itself after fixed interval of time called fundamental period.

→ $x[n] = x[n+N]$, it must be satisfied where,

N → fundamental period

→ The periodic signal are all power signal

→ The power signal are band unlimited.

Non-Periodic

→ They do not repeat after fixed interval of time

→ They does not satisfied $x[n] = x[n+N]$

→ The non-periodic signal are mostly Energy signal

→ They are band limited signal

Imp

$$x[n] = x[-n]$$

$$x[n] = -x[-n]$$

Q. Derive a condition for periodicity of the discrete time sinusoidal signal

Or, Prove that sinusoidal signal will be periodic, if its frequency is rational.

⇒ Consider a discrete time sinusoidal signal $x[n] = A \sin(\omega_0 n + \phi)$, Where ϕ is the phase difference

We have,

$$x[n] = x[n+N], \text{ Where } N \text{ is the fundamental period.}$$

$$\text{Now, } x[n] = A \sin[\omega_0(n+N) + \phi] \\ = A \sin[\omega_0 n + \phi + \omega_0 N] \quad \text{--- (ii)}$$

then eqn (i) and (ii) are equal only if $\omega_0 N$ is the multiple of 2π .

i.e

$$\omega_0 N = 2\pi k \quad \text{Where } k \text{ is any integer}$$

$$2\pi f N = 2\pi k$$

$\therefore f = k/N$ Since the frequency is the ratio of integer and fund.p.
So, the frequency is rational.

A discrete time sinusoidal signal will be periodic only if its frequency is rational.

Q.7] check whether the given signals are periodic or non-periodic. If periodic determine the fundamental period.

a) $x[n] = 4\sin\left(\frac{3}{5}\pi n\right)$

b) $x[n] = 5\cos(n/8)$

c) $x[n] = 4\sin\left(\frac{3}{5}\pi n\right) + 5\cos\left(\frac{1}{3}\pi n\right)$

d) $x[n] = 2e^{-j(3/8\pi n + \pi/4)}$

e) $x[n] = 2\cos\left(\frac{3}{7}\pi n + \frac{3}{4}\pi\right)$

a) $x[n] = 4\sin\left(\frac{3}{5}\pi n\right)$

Comparing with $x[n] = A\sin\omega n$
we get,

$$\omega n = \frac{3}{5}\pi n$$

$$2\pi f n = \frac{3}{5}\pi n$$

$$f = \frac{3}{10} = \frac{k}{N}$$

Since the frequency is rational then signal is periodic with $N = 10$.

b) $x[n] = 5\cos(n/8)$

Comparing with $x[n] = A\cos\omega n$
we get,

$$\omega n = n/8$$

$$2\pi f n = n/8$$

$f = \frac{1}{16\pi}$ Since the frequency is irrational so, it is non periodic signal.

c) $x[n] = 4\sin\left(\frac{3}{5}\pi n\right) + 5\cos\left(\frac{1}{3}\pi n\right)$

Comparing with $x[n] = A\sin\left(\frac{3}{5}\pi n\right) + B\sin\left(\frac{1}{3}\pi n\right)$
we get,

$$\omega_1 n = \frac{3}{5}\pi n$$

$$2\pi f_1 n = \frac{3}{5}\pi n$$

$$f_1 = \frac{3}{10} = \frac{k}{N_1}$$

Again,

$$\omega_2 n = \frac{1}{3}\pi n$$

$$2\pi f_2 n = \frac{1}{3}\pi n$$

$$f_2 = \frac{1}{6} = \frac{k}{N_2}$$

here, $N_1 = 10$ and $N_2 = 6$

which is periodic since the frequency is rational with N_1 and N_2 .

So, fundamental period = common remaining (L.C.M)
 $= 2 \times (5 \times 3)$
 $= 30$ #

$$d) x[n] = A \left[e^{-j \left(\frac{3}{8} \pi n + \frac{\pi}{4} \right)} + j \sin \left(\frac{3}{8} \pi n + \frac{\pi}{4} \right) \right]$$

$$= 2 \left[\cos \left(\frac{3}{8} \pi n + \frac{\pi}{4} \right) + j \sin \left(\frac{3}{8} \pi n + \frac{\pi}{4} \right) \right]$$

Comparing with $x[n] = A [\cos(\omega n + \phi) + j \sin(\omega n + \phi)]$

we get,

$$\omega n = \frac{3}{8} \pi n$$

$$2\pi f n = \frac{3}{8} \pi n$$

$$f = \frac{3}{16} = \frac{K}{N}, \text{ which is periodic}$$

Since, the frequency is rational

now, fundamental period = 16 #

1) Energy and Power Signal.

Energy Signal	Power Signal
→ Energy signal have finity energy and zero average power.	→ Power signal have finite power and infinite energy
→ it's non-periodic in nature.	→ It is periodic in nature.
→ Energy signal are band limited signal	→ Power signal are band unlimited signal
→ $x[n]$ is energy signal with energy.	→ The signal $x[n]$ is Power signal with power.
$E = \sum_{n=-\infty}^{\infty} x[n] ^2$	$P_{\text{average}} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] ^2$
	Where N fundamental period.

Note: $n+1 = N$
 $n = N-1$

$|\cos \phi| = 1$, max^m value
 $|\sin \phi| = 1$, max^m value

Q1 Check whether the signal is energy or power signal, determine the energy and power of the given signal

a) $x[n] = 4 \cos\left(\frac{3}{4}\pi n + \phi\right)$

b) $x[n] = 4 \{ 4[n] - 4[n-4] \}$

a) $x[n] = 4 \cos\left(\frac{3}{4}\pi n + \phi\right)$
Comparing with $x[n] = A \cos(\omega n + \phi)$
We get,

$$\omega n = \frac{3}{4}\pi n$$

$$2\pi f n = \frac{3}{4}\pi n$$

$$f = \frac{3}{8} = \frac{k}{N}, \text{ which is rational so, it is periodic.}$$

Now,

$$N = 8$$

$$P_{avg} = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

$$= \frac{1}{8} \sum_{n=0}^7 \left| 4 \cos\left(\frac{3}{4}\pi n + \phi\right) \right|^2$$

$$= \frac{16}{8} \sum_{n=0}^7 \left| \cos\left(\frac{3}{4}\pi n + \phi\right) \right|^2$$

$$= 2 \sum_{n=0}^7 1$$

$$= 2(1+1+1+1+1+1+1+1) = 16$$

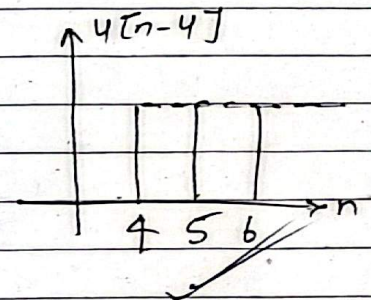
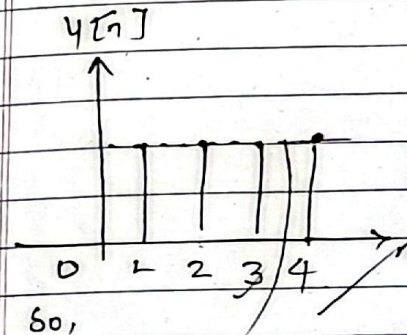
b) $x[n] = 4 \{ 4[n] - 4[n-4] \}$
We know,

$$4[n] = 1; \text{ for } n \geq 0$$

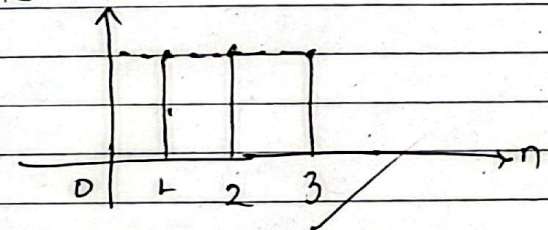
$$0; \text{ otherwise}$$

$$4[n-4] = 1; \text{ for } n-4 \geq 0 \text{ i.e. } n \geq 4$$

$$0; \text{ otherwise}$$



So, $4[n] - 4[n-4]$



Now, $x[n] = 4$; for $0 \leq n \leq 3$

$$E = \sum_{n=0}^3 |4|^2 = \sum_{n=0}^3 16$$

$$= (16 + 16 + 16 + 16)$$

$$= 64$$

c) $x[n] = r^n u[n]$; for $r < 1$

d) $x[n] = r^n \cdot u[n - n_0]$; for $r < 1$

c) Since, the signal is non periodic. So, it is energy signal.

We know, $u[n] = 1$; for $n \geq 0$
0 ; otherwise

Now,
Energy (E) = $\sum_{n=-\infty}^{\infty} |x[n]|^2$

= $\sum_{n=0}^{\infty} |r^n \cdot 1|^2$

= $\sum_{n=0}^{\infty} |r^2|^n$

= $1 + (r^2)^1 + (r^2)^2 + (r^2)^3 + (r^2)^4 + \dots$

= $\frac{1}{1 - r^2}$ $\left[\frac{a}{1 - r} \right]$ Where $a \rightarrow$ first term
$r =$ Common ratio.

d) $x[n] = r^n u[n - n_0]$

We know,

$u[n - n_0] = 1$; for $n - n_0 \geq 0$ i.e. $n \geq n_0$

Which is non-periodic signal so it is energy signal.

Now,

(E) = $\sum_{n=-\infty}^{\infty} |x[n]|^2$

= $\sum_{n=n_0}^{\infty} |r^n|^2$

= $\sum_{n=n_0}^{\infty} |r^2|^n$

= $r^{2n_0} + r^{2n_0+2} + r^{2n_0+4} + r^{2n_0+6} + \dots$

= $r^{2n_0} (1 + r^2 + r^4 + r^6 + \dots)$

= $r^{2n_0} (1 + (r^2)^1 + (r^2)^2 + (r^2)^3 + \dots)$

= $\frac{r^{2n_0}}{1 - r^2}$ #

e) $x[n] = 2^n [u[n] - u[n-10]]$

We know,

$$u[n] = 1; \text{ for } n \geq 0$$

$$0; \text{ otherwise.}$$

$$u[n-10] = 1; \text{ for } n-10 \geq 0 \text{ i.e. } n \geq 10$$

$$0; \text{ otherwise.}$$

Range $\rightarrow x[n] = 2^n; 0 \leq n \leq 9$

which is non-periodic, so it is energy signal

$$(E) = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_0^9 |2^n|^2$$

$$= 1 + 2^2 + 2^4 + 2^6 + 2^8 + \dots + 2^{18}$$

$$= [1 + 2^2 + (2^2)^2 + (2^2)^3 + \dots + (2^2)^9]$$

$$= \frac{1 - (2^2)^{10}}{1 - 2^2}$$

$$= \frac{1 - 2^{20}}{1 - 2^2} \quad \#$$

$$\frac{1 - ar^n}{1 - r}$$

d) Deterministic signal

$\rightarrow$ Those signal whose occurrence can be predicted in advance are called Deterministic signal

$\rightarrow$ They are regular or occurrence.

$\rightarrow$ They can express mathematically

$\rightarrow$ They are regular signal
e.g. sinusoidal signal

Non-Deterministic (Random)

$\rightarrow$ Those signal whose occurrence can't be predicted in advance are called Non-Deterministic signal

$\rightarrow$ They are Random

$\rightarrow$ They can not be express mathematically

$\rightarrow$ They are random signal

Eg. noise signal.

Signal -
↳ Carries certain information.

* Types of signal on the basis of time domain

1) Analog signal (Continuous, Time domain signal)

Eg. $x(t)$, $g(t)$, $f(t)$

t → Continuous time domain representation.

2) Digital signal (Discrete time domain signal)

Eg. $x[n]$, $y[n]$, $g[n]$

n → Discrete time domain representation
(with square pulses)

Q.1] Sketch the following signals.

$$x(t) = 2 \{ u(t+2) - u(t-2) \}$$

Where, $u(t)$ → unit step signal

$$\therefore u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for Continuous time domain}$$

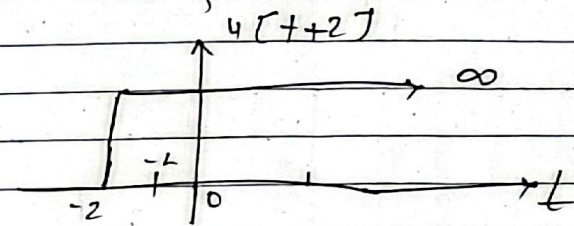
$$\therefore u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for Discrete time domain}$$

Solⁿ:- $x(t) = 2 \{ u(t+2) - u(t-2) \}$

Here,

$$u(t+2) = 1; \text{ for } t+2 \geq 0 \text{ i.e. } t \geq -2$$

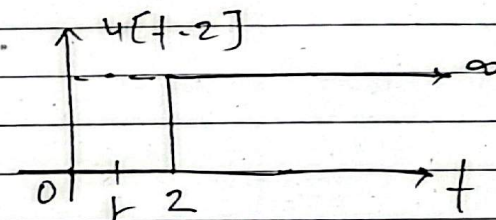
$$= 0; \text{ otherwise.}$$



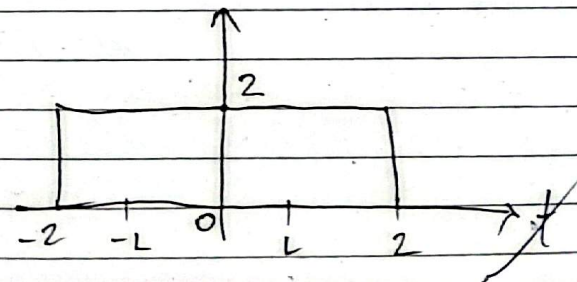
Again,

$$u(t-2) = 1; \text{ for } t-2 \geq 0 \text{ i.e. } t \geq 2$$

$$= 0; \text{ otherwise.}$$



Now, $x(t) = 2 \{ u(t+2) - u(t-2) \}$
↳ Amplitude.

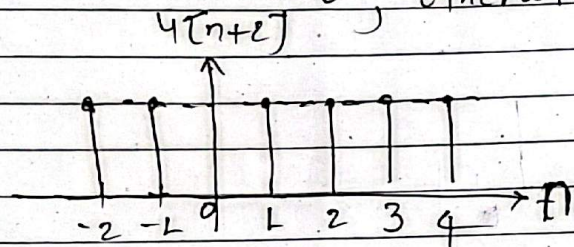


Q.107 Plot,

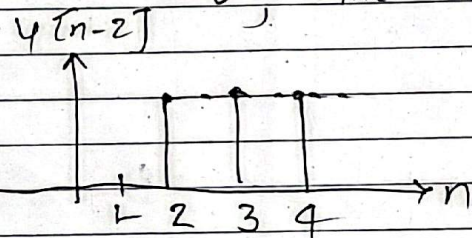
$$x[n] = 2[u[n+2] - u[n-2]]$$

Here,

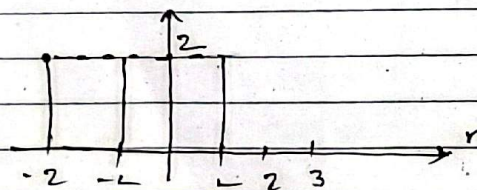
$$u[n+2] = \begin{cases} 1 & \text{for } n+2 \geq 0 \text{ i.e. } n \geq -2 \\ 0 & \text{otherwise.} \end{cases}$$



$$\text{Again, } u[n-2] = \begin{cases} 1 & \text{for } n-2 \geq 0 \text{ i.e. } n \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$



$$\text{Now, } x[n] = 2[u[n+2] - u[n-2]]$$



$$\text{Here, } x[-2] = 2$$

$$x[-1] = 2$$

$$x[0] = 2$$

$$x[1] = 2$$

$$x[2] = 0$$

Also,

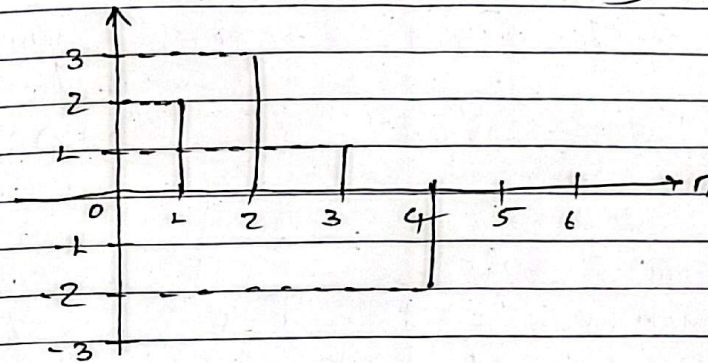
$$x[n] = \{2, 2, 2, 2\} \quad \text{for } -2 \leq n \leq 1$$

$$0 \text{ : otherwise}$$

Plot,

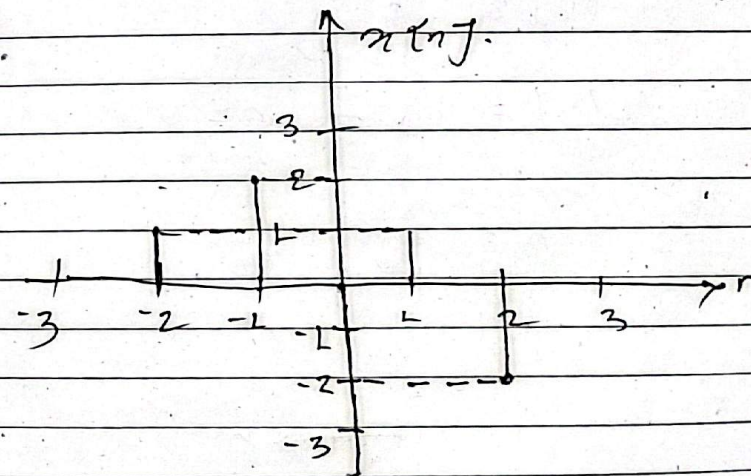
$$\text{Q.108] } x[n] = \{1, 2, 3, 1, -2\}$$

↑
origin (some time bolded)



$$\text{Q.109] Plot, } x[n] = \{1, 2, -1, 1, -2\}$$

$$\text{When, } n = \{-2, -1, 0, 1, 2\}$$



Q. N] Represent Graphically.

(a) $x[n] = u[n]$

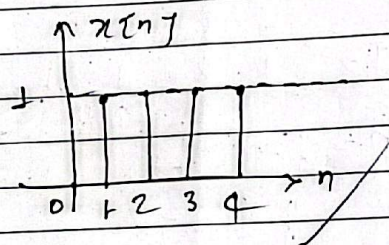
(b) $x[n] = \delta[n]$; where $\delta[n]$ is unit impulse signal

(c) $x[n] = \sum_{i=0}^{\infty} \delta[n-i]$

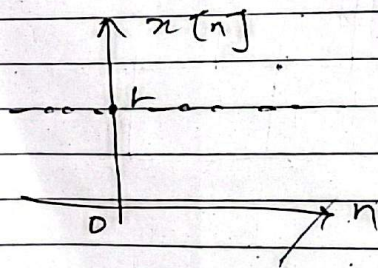
$\delta[n] = 1$; for $n = 0$
0; otherwise.

So $x[n] = u[n]$

i.e. $u[n] = 1$; for $n \geq 0$
0; otherwise.



(b) $x[n] = \delta[n] = 1$; for $n = 0$
0; otherwise.



(c) $x[n] = \sum_{i=0}^{\infty} \delta[n-i]$

for $i=0$

$x[n] = \delta[n]$

for $i=1$,

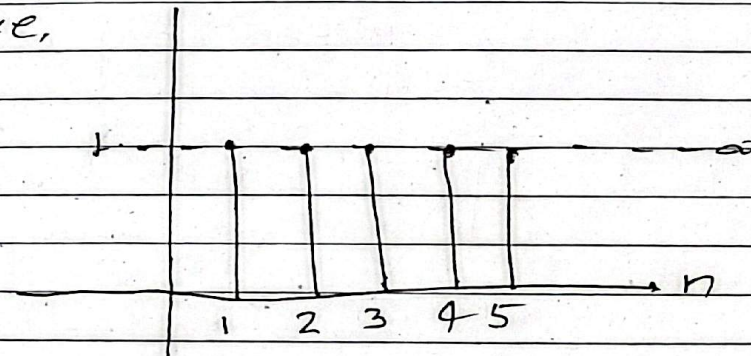
$x[n] = \delta[n-1] = 1$; for $n-1=0$ i.e. $n=1$
0; otherwise.

for $i=2$,

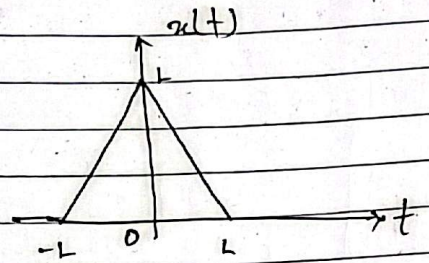
$x[n] = \delta[n-2] = 1$; for $n-2=0$ i.e. $n=2$
0; otherwise.

So, on.

Hence,



Q.P.] Given $x(t)$ as

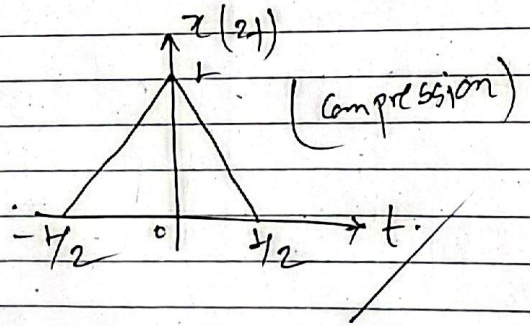


Sketch,

- (a) $x(2t)$
- (b) $x(t/2)$
- (c) $x(t+2)$
- (d) $x(t-2)$

soln:

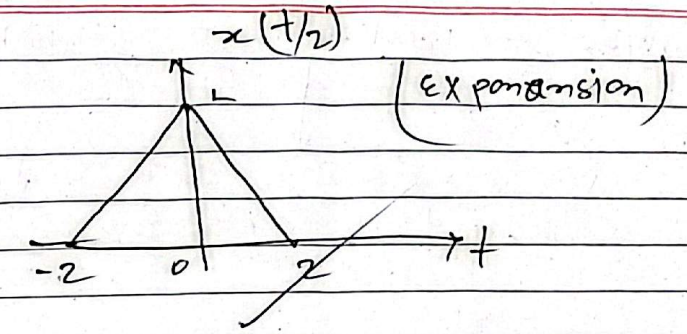
(a) $x(2t) = 1$ for, $2t = t$ i.e. $t = 1/2$
 $2t = -t$ i.e. $t = -1/2$



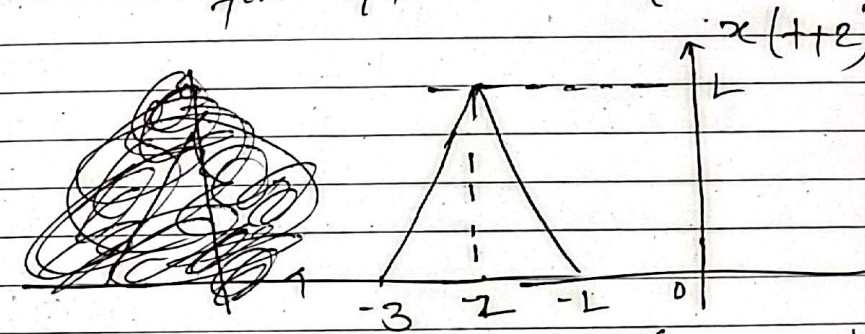
(b)

$x(t/2)$.
 $x(t/2) = 0$ for $t/2 = -1 \therefore t = -2$
 $= 0$ for, $t/2 = 1 \therefore t = 2$
 $= 1$ for, $t/2 = 0 \therefore t = 0$

then,

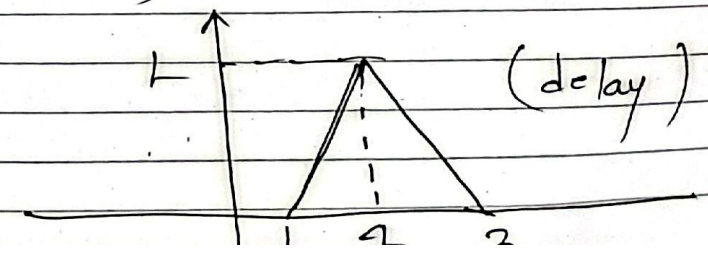


(c) $x(t+2) = 0$ for $t+2 = -1 \therefore t = -3$
 $= 0$ for, $t+2 = 1 \therefore t = -1$
 $= 1$ for, $t+2 = 0 \therefore t = -2$



(d)

$x(t-2) = 0$ for $t-2 = 1$ i.e. $t = 3$
 $= 0$; $t-2 = -1$ i.e. $t = 1$
 $= 1$; $t-2 = 0$ i.e. $t = 2$



System :- Based on signal on which they operate

continuous time system

Discrete time system

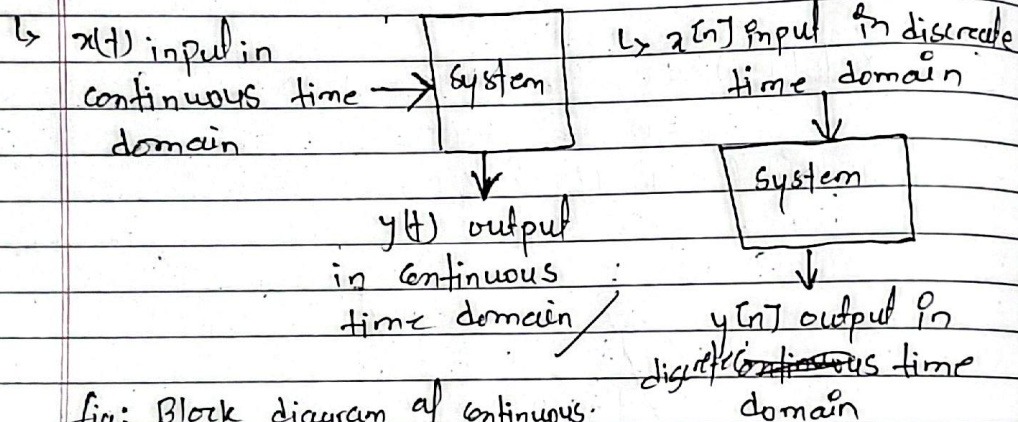


fig: Block diagram of continuous time system

fig: Block diagram of discrete time system

$$y(t) = \int \xi x(t) dt$$

Where,

$x(t) / x[n] \rightarrow$ I/p signal

$y(t) / y[n] \rightarrow$ O/p signal

$$y[n] = \sum \xi x[n]$$

Properties of system

- ↳ Linearity ✓
- ↳ Causality ✓
- ↳ Stability ✓
- ↳ Static / dynamic ✓
- ↳ Time Invariance ✓
- ↳ memory ✓

Q.] checks the system properties :-

- a) $y[n] = A x[n] \cos \omega n$
- b) $y[n] = A x^2[n]$
- c) $y[n] = A x[n^2]$
- d) $y[n] = A x[2n]$
- e) $y[n] = A x[n] + B$
- f) $y[n] = x[n] + \frac{3}{4} x[n-1]$
- g) $y[n] = e^{-x[n]}$

b) so $y[n] = A x^2[n]$

for linearity,
 $\alpha \cdot \sum \xi x[n] = \sum \xi \alpha \cdot x[n]$

Taking,
 $\alpha \cdot \sum \xi x[n] = \alpha \cdot y[n]$
 $= \alpha \cdot A x^2[n] \quad \text{--- (i)}$

Again $\int \{ \alpha \cdot x[n] \}^2 = A \int \{ \alpha \cdot x[n] \}^2$
 $= A \alpha^2 \int x^2[n] \quad \text{--- (ii)}$

from eqn (i) and (ii),
 $\alpha \cdot \int x^2[n] \neq \int \{ \alpha \cdot x[n] \}^2$

So, the given system is non-linear.

for causality $\rightarrow$ [Past and present Output / input]

$\rightarrow$ if it's future output then non-causal
 $y[n] = A x[n]$

When $n=0$; $y[0] = A x[0]$

$n=1$; $y[1] = A x[1]$

$n=2$; $y[2] = A x[2]$

Since, the output $y[n]$ depends on the present value of input $x[n]$. So, the system is causal.

for static / dynamic $\rightarrow$ present value depends then static

$y[n] = A x[n]$

$\rightarrow$ past / future value depends then dynamic

Since, the value of $y[n]$ depends on present input $x[n]$ then the system is stable.

for stability, [BIBO stable]

let the i/p be bounded.

$x[n] \leq M_x < \infty$

i.e $y[n] = |x[n]|$

where,

M_x any finite value.

$< \infty$

then the corresponding o/p

$y[n] = |A x[n]|$
 $\leq |A| |x[n]|^2$

$\leq |A| \cdot M_x^2 < \infty$

Since, the o/p $y[n]$ is bounded. Hence the system is stable.

for Time Invariance:-

$y[n] = A x[n]$

$y[n, n_0] = y[n - n_0]$ for Time Invariant

where,

$y[n, n_0] = \text{o/p}$ when only i/p is shifted by 'no' quantity.

$y[n - n_0] = \text{o/p}$ when o/p is shifted by 'no' quantity

When the i/p is shifted by 'no' quantity
 $x[n] \rightarrow x[n-n_0]$ the corresponding
 o/p is:
 $y[n, n_0] = A x^2[n-n_0]$ (i)

Again, the o/p is shifted by 'no' quantity
 $y[n] \rightarrow y[n-n_0] = A x^2[n-n_0]$ (ii)

from eqn (i) and (ii),
 $y[n, n_0] = y[n-n_0]$

Hence, the system is time invariant

a) Soln: $y[n] = A x[n] \cos \omega_0 n$
 for linearity, $\alpha \cdot \int x[n] \cdot y[n] = \int \alpha \cdot x[n] \cdot y[n]$
 taking,
 $\alpha \cdot \int x[n] \cdot y[n] = \alpha \cdot y[n]$
 $= \alpha \cdot A x[n] \cos \omega_0 n$ (i)

Again,
 $\int \alpha x[n] \cdot y[n] = A \int \alpha x[n] \cos \omega_0 n$
 $= \alpha \cdot A x[n] \cos \omega_0 n$ (ii)

from eqn (i) and (ii), are same then the system
 is linear.

for Causality, $y[n] = A x[n] \cos \omega_0 n$
 When, $n=0$: $y[0] = A x[0] \cos 0$
 $n=1$: $y[1] = A x[1] \cos \omega_0$
 $n=2$: $y[2] = A x[2] \cos 2\omega_0$

Since, the o/p $y[n]$ depends on the present
 value of i/p $x[n]$ then the system is causal.

for static/dynamic, $y[n] = A x[n] \cos \omega_0 n$.

Since, the value $y[n]$ depends on present
 input $x[n]$ then the system is static.

for stability,
 Let the i/p be bounded, $x[n] \leq Mx$.
 then corresponding o/p is,

$$y[n] = |A x[n] \cos \omega_0 n|$$

$$\leq |A| \cdot Mx \cdot 1 < \infty \quad [\because \cos \omega_0 n = 1 \text{ max value}]$$

$$\leq |A| \cdot Mx < \infty$$

Since, the o/p $y[n]$ is bounded. the the system
 is stable.

for Time Invariant;
 When the i/p is shifted by 'no' quantity
 $x[n] \rightarrow x[n-n_0]$ the corresponding o/p is
 $y[n, n_0] = A x[n-n_0] \cos \omega_0 n$ (i)

Again, When o/p is shifted by 'no' quantity.
 $y[n] \rightarrow y[n-n_0] = A x[n-n_0] \cdot \cos \omega_0 (n-n_0)$ (ii)

from (i) and (ii),

$$y[n, n_0] \neq y[n-n_0]$$

hence, the system is variant.

d) Solⁿ: $y[n] = A x[n]$
 for linearity, taking $\alpha \cdot \{x[n]\} = \alpha \cdot y[n]$
 $= \alpha \cdot A x[n]$ — (i)
 taking, $\{ \alpha \cdot x[n] \} = A \{ \alpha \cdot x[n] \}$
 $= \alpha \cdot A x[n]$ — (ii)

from (i) and (ii) are same, so the system is linear.

for Causality, $y[n] = A x[n]$

When $n=0$: $y[0] = A x[0]$

$n=1$: $y[1] = A x[1]$

$n=2$: $y[2] = A x[4]$

Since, the o/p depends on the future i/p so, the system is non-causal.

for Static/dynamic, $y[n] = A x[n]$

Since, the value of $y[n]$ depends on future i/p $x[n]$ so, the system is dynamic.

for Stability, let i/p is bounded then the corresponding o/p is

$$y[n] = |A x[n]|$$

$$\leq |A| |x[n]|$$

$$\leq |A| \cdot M_x < \infty$$

hence, the system is stable.

for Time Invariants,

When the i/p is shifted 'no' quantity.
 $x[n] \rightarrow x[n-n_0]$ the corresponding o/p is

$$y[n, n_0] = A x[n(n-n_0)]$$

Again, when o/p is shifted by 'no' quantity.

$$y[n] \rightarrow y[n-n_0] = A x[n(n-n_0)] \text{ — (ii)}$$

hence,

$$y[n, n_0] = y[n-n_0]$$

the system is time Invariant.

Solⁿ: $y[n] = x[n] + \frac{3}{4} x[n-1]$

for stability, let i/p be bounded.

$$\text{i.e. } x[n] \leq M_x < \infty$$

$$\text{i.e. } x[n-1] \leq M_x < \infty$$

then the corresponding o/p is,

$$y[n] = |x[n] + \frac{3}{4} x[n-1]|$$

$$\leq |x[n]| + \left| \frac{3}{4} \right| |x[n-1]|$$

$$\leq M_x + \left| \frac{3}{4} \right| \cdot M_x < \infty$$

Hence, the system is stable.

for time Invariant,

$$y[n] = x[n] + \frac{3}{4} x[n-1]$$

When the I/p is shifted by 'no' quantity,
 $x[n] \rightarrow x[n-no]$ then the corresponding
 o/p is,

$$y[n, no] = x[n-no] + \frac{3}{4} x[n-no-1] \quad \text{--- (i)}$$

Again,

When o/p is shifted by 'no' quantity,
 $y[n] \rightarrow y[n-no] = x[n-no] + \frac{3}{4} x[n-no-1]$ --- (ii)

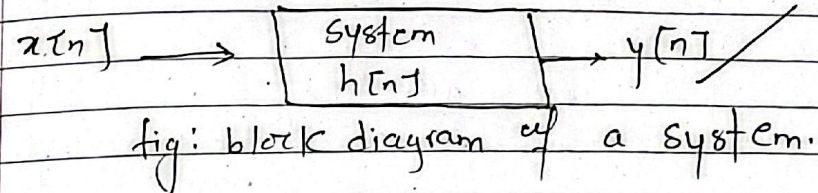
from eqⁿ (i) and (ii),

$$y[n, no] = y[n-no]$$

Hence, the system is time Invariant.

V.V.V. Imrf fix question!

Linear Convolution [Convolution Sum]
 LTI System.



LTI $\rightarrow$ linear time Invariant System.

$x[n] \rightarrow$ I/p
 $y[n] \rightarrow$ o/p $\rightarrow$ Response / System Response.

$h[n] \rightarrow$ Impulse Response. $h[n] \rightarrow$

$\rightarrow$ The Linear Convolution state that,
 $y[n] = x[n] \otimes h[n]$
 $\uparrow$ convolution symbol.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

Convolution Sum (linear Convolution)

$\hookrightarrow$ Gives the relation betⁿ I/p & o/p.

$\rightarrow$ If $x[n]$ has length L_1
 If $h[n]$ has length L_2
 then $y[n]$ will have length, $L = L_1 + L_2 - 1$

↳ If $x[n]$ has N_1 number of value in -ve 'n'

↳ $h[n]$ has N_2 " " " " -ve 'n'

then

$y[n]$ will have
 $N_1 + N_2$ No. of values in -ve 'n'.

Q.7. Determine the response of LTI System having
 Input $x[n] = \{1, 1, 1\}$ and Impulse response
 $h[n] = \{1, 2, 1\}$

⇒ Given,

length of $x[n]$; $L_1 = 3$

length of $h[n]$; $L_2 = 3$

then

length of $y[n]$, $L = L_1 + L_2 - 1$
 $= 3 + 3 - 1$
 $= 5$

Note: $x[n] = \{1, 1, 1\}$ mean $x[0] = 1, x[1] = 1, x[2] = 1$

$h[n] = \{1, 2, 1\}$, means $h[-1] = 1, h[0] = 2, h[1] = 1$

↑ origin represent
 Arrow sign denote origin

We know by Convolution.

$$y[n] = x[n] \otimes h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$\text{or, } y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$= x[0] \cdot h[n] + x[1] \cdot h[n-1] + x[2] \cdot h[n-2]$$

$$\therefore y[n] = h[n] + h[n-1] + h[n-2]$$

$$\text{When, } y[-1] = h[-1] + h[-2] + h[-3]$$

$$n=-1, \quad = 1 + 0 + 0$$

$$= 1$$

$$\text{When } n=0, y[0] = h[0] + h[-1] + h[-2]$$

$$= 2 + 1 + 0$$

$$= 3$$

$$\text{When } n=1, y[1] = h[1] + h[0] + h[-1]$$

$$= 1 + 2 + 1$$

$$= 4$$

$$\text{When } n=2, y[2] = h[2] + h[1] + h[0]$$

$$= 0 + 1 + 2$$

$$= 3$$

When $n=3$, $y[3] = h[3] + h[2] + h[1]$
 $= 0 + 0 + 1$
 $= 1$

Hence,

required solⁿ is $y[n] = \{1, 3, 4, 3, 1\}$

for Answer checks,

		$y[n-1]$	$y[n]$	$y[n+1]$
$h[n]$	1	1	1	1
	2	2	2	2
	1	1	1	1

No need.

2.8] $x[n] = a^n u[n]$
 $h[n] = u[n]$
 $y[n] = ?$

Solⁿ: We know by convolution sum,
 $y[n] = x[n] \otimes h[n]$
 $= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$

where,

$x[k] = a^k u[k]$

$= a^k \cdot 1$; for $k \geq 0$
 0 ; otherwise.

$h[n-k] = u[n-k] = 1$; for $n-k \geq 0$ i.e. $k \leq n$
 0 ; otherwise.

Now,
 $y[n] = \sum_{k=0}^n a^k \cdot 1$

$= \sum_{k=0}^n a^k$

$= 1 + a + a^2 + a^3 + \dots + a^n$

$= \frac{1 - a^{n+1}}{1 - a}$ $\left[\because y = \frac{a(1 - r^{n+1})}{1 - r} \right]$
 where $a \rightarrow$ first term
 $r =$ Common ratio

3.8] $x[n] = u[n]$
 $h[n] = \lambda^n u[n - n_0]$
 $y[n] = ?$

Solⁿ:-

We know,

$y[n] = x[n] \otimes h[n]$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] \quad \text{--- (1)}$$

Where,

$$x[k] = u[k] = 1; \text{ for } k \geq 0 \\ 0; \text{ otherwise}$$

$$h[n-k] = a^{n-k} u[n-k-n_0] = 1; \text{ for } n-k-n_0 \geq 0 \\ \text{i.e. } k = n-n_0 \\ 0; \text{ otherwise}$$

from eqn (1),

$$y[n] = \sum_{k=0}^{n-n_0} a^{n-k} \cdot 1$$

$$= \sum_{k=0}^{n-n_0} a^n \cdot a^{-k}$$

$$= a^n \sum_{k=0}^{n-n_0} \left(\frac{1}{a}\right)^k$$

$$= a^n \left[1 + \left(\frac{1}{a}\right)^1 + \left(\frac{1}{a}\right)^2 + \dots + \left(\frac{1}{a}\right)^{n-n_0} \right]$$

$$= a^n \left[\frac{1 - \left(\frac{1}{a}\right)^{n-n_0+1}}{1 - \frac{1}{a}} \right] \quad \#$$

$$Q.7] \quad x[n] = \{u[n] - u[n-10]\}$$

$$h[n] = a^n u[n]$$

$$y[n] = ?$$

Soln:

We know

$$y[n] = x[n] \otimes h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] \quad \text{--- (1)}$$

Where,

$$x[k] = \{u[k] - u[k-10]\}$$

$$\text{When, } u[k] = 1; \text{ for } k \geq 0 \\ 0; \text{ otherwise}$$

$$u[k-10] = 1; \text{ for } k-10 \geq 0 \\ \text{i.e. } k \geq 10 \\ 0; \text{ otherwise}$$

$$\text{So, } x[k] = 1 \text{ for } 0 \leq k \leq 9$$

Again,

$$h[n-k] = a^{n-k} u[n-k] = \begin{cases} a^{n-k} & \text{for } n-k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $k \leq n$

Now,

$$y[n] = \sum_{k=0}^n 1 \cdot a^{n-k} \quad ; \text{ for } n \leq 9$$

$$= \sum_{k=0}^n a^n \left(\frac{1}{a}\right)^k$$

$$= a^n \left[1 + \left(\frac{1}{a}\right) + \left(\frac{1}{a}\right)^2 + \left(\frac{1}{a}\right)^3 + \dots + \left(\frac{1}{a}\right)^n \right]$$

$$= a^n \left[\frac{1 - \left(\frac{1}{a}\right)^{n+1}}{1 - \left(\frac{1}{a}\right)} \right]$$

[Handwritten signature]

Q7. Determine the type of an LTI system.
Whose impulse response $h[n] = 2, 1, 2, 1, 2$

have an output sequence $y[n] = 9, 2, 1, 3, 2, 1, 6$

$$\Rightarrow \begin{aligned} \text{Let } x[n] &= L_1 \\ h[n] &= L_2 = 3 \\ y[n] &= L = 6 \end{aligned}$$

We know,

$$\begin{aligned} L &= L_1 + L_2 - L_1 \\ L_1 &= L + L_2 - L_2 \end{aligned}$$

$$= 6 + 1 - 3 = 4$$

length of $x[n]$

By Convolution solution,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$= \sum_{k=-L}^L x[k] \cdot h[n-k]$$

$$= x[-1] h[n+1] + x[0] h[n] + x[1] h[n-1] + x[2] h[n-2]$$

When,

$$n = -2, \quad y[-2] = x[-1]h[-1] + x[0]h[-2] + x[1]h[-3] \\ + x[2]h[-4]$$

$$2 = x[-1] + 0 + 0 + 0$$

$$\therefore x[-1] = 2$$

When

$$n = -1, \quad y[-1] = x[-1]h[0] + x[0]h[-1] + x[1]h[-2] \\ + x[2]h[-3]$$

$$1 = 2 \cdot 2 + x[0] \cdot 2 + 0 + 0$$

$$x[0] = -3$$

When,

$$n = 0, \quad y[0] = x[-1]h[1] + x[0]h[0] + x[1]h[-1] \\ + x[2]h[-2]$$

$$3 = 2 \cdot 1 + (-3) \cdot 2 + x[1] \cdot 1 + 0$$

$$\therefore x[1] = 7$$

When

$$n = 1, \quad y[1] = 2 \cdot 0 + (-3) \cdot 1 + 7 \cdot 2 + x[2] \cdot 1$$

$$2 = -3 + 14 + x[2]$$

$$\therefore x[2] = -9$$

Hence,

required ~~q/p~~ is $x[n] = \{2, -3, 7, -9\}$

↑

h

Convolution by Graphical method,

we know,

$$y[n] = x[n] \otimes h[n]$$

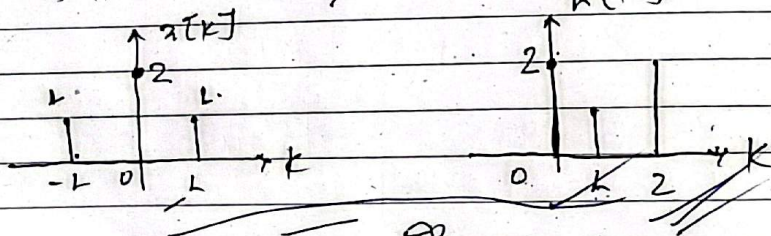
$$= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

Eg. $x[n] = \{1, 2, 1\}$ length $(L_1) = 3$

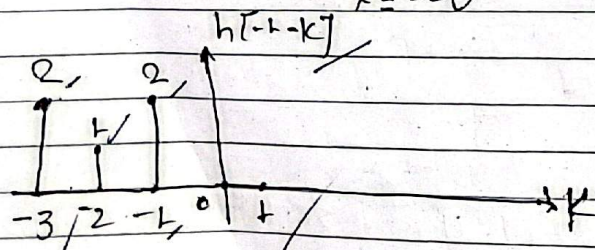
& $h[n] = \{2, 1, 2\}$ length $(L_2) = 3$

Hence, length $(L) = L_1 + L_2 - 1 = 5$

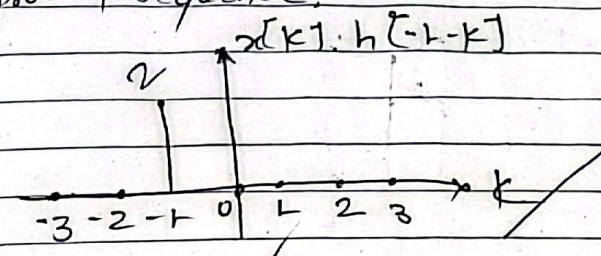
for graphical method,



When, $n=-1$; $y[-1] = \sum_{k=-\infty}^{\infty} x[k] h[-1-k]$

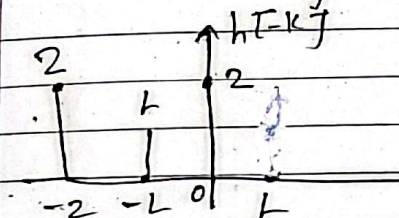


product sequence,

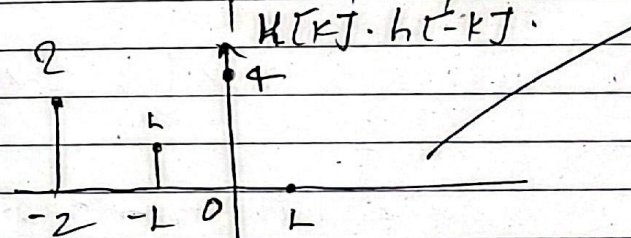


$\therefore y[-1] = 2$

When, $n=0$, $y[0] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[-k]$

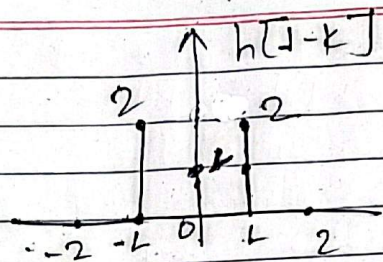


product sequence,



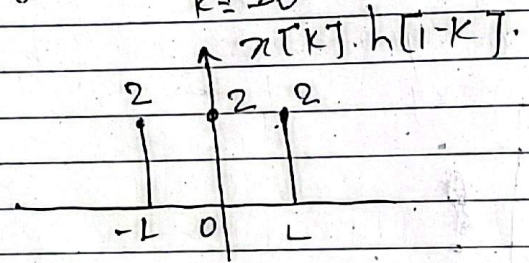
$\therefore y[0] = 1 + 4 = 5$

When $n=1$, $y[1] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[1-k]$



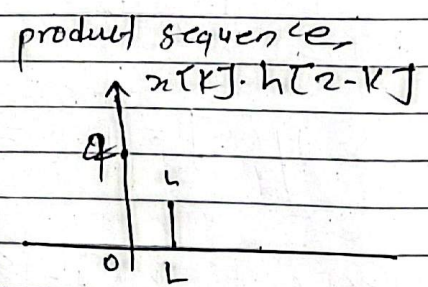
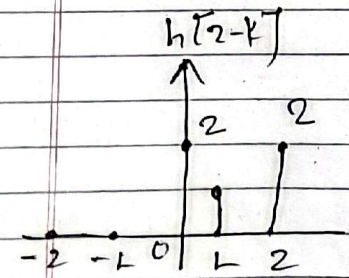
product sequence,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[1-k]$$



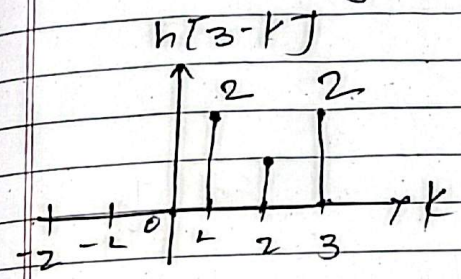
$\therefore y[1] = 6$

When, $n=2$, $y[2] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[2-k]$

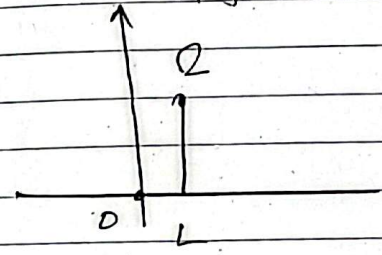


$\therefore y[2] = 4 + 1 = 5$

When, $n=3$, $y[3] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[3-k]$



product sequence,
 $x[k] \cdot h[3-k]$



$\therefore y[3] = 2$

The required solⁿ is,

$y[n] = \{ 2, 5, 6, 5, 2 \}$

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Properties:-
properties of convolution (sum):

① Commutative property.

It state that,

$$y[n] = x[n] \otimes h[n] = h[n] \otimes x[n]$$

$$\therefore y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

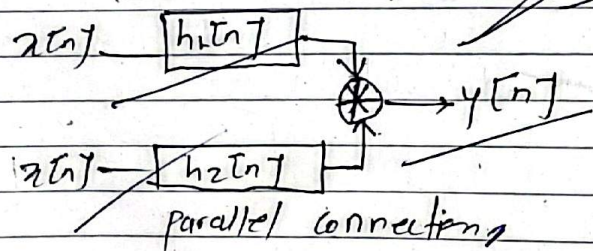
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② Distributive property.

It state that,

$$y[n] = x[n] \otimes \{h_1[n] + h_2[n]\}$$

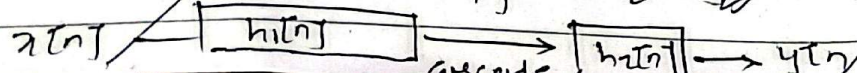
$$= x[n] \otimes h_1[n] + x[n] \otimes h_2[n]$$



③ Associative Property, It state that,

$$y[n] = x[n] \otimes \{h_1[n] \otimes h_2[n]\}$$

$$= \{x[n] \otimes h_1[n]\} \otimes h_2[n]$$



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Q.1 verify using commutative property of convolution for the signal $x[n] = \{1, 2, 1, 2\}$ and $h[n] = \{2, 2, -1, 2\}$ also sketch the given signal and the response graphically.

Q.2. Two Subsystem have cascaded with each other. determine the response of the system. if the subsystem are identically with the impulse response. $h_1[n] = h_2[n] = h[n] = \{1, 2, 2\}$. The system is excited by input $x[n] = 4[n+2] - 4[n-2]$

Q.N.3] Two Subsystem are connected in parallel connection with impulse response $h_1[n] = \{1, 2, 3\}$ & $h_2[n] = \{2, 1, 2\}$. If the system has input $x[n] = \{1, 1, 1, 2\}$. Determine the output response of the system.

Q. Necessary and sufficient condition for an LTI system to be:
(i) causal and (ii) stable.

(i) Causality:-

We know by convolution sum

$$y[n] = x[n] \otimes h[n]$$

$$\therefore y[n] = h[n] \otimes x[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

So,

$$y[n] = \underbrace{\sum_{k=-\infty}^{-1} h[k] \cdot x[n-k]}_{1^{st} \text{ part}} + \underbrace{\sum_{k=0}^{\infty} h[k] \cdot x[n-k]}_{2^{nd} \text{ part}} \quad \text{--- (i)}$$

$x[n]$ → Present I/p
 $x[n-k]$ → Past I/p
 $x[n+k]$ → future I/p

from eqn (i), It is clear that in the first part the input $x[n-k]$ corresponds to future value as k is in negative. Hence, for the system to be causal the first part must be zero.
 Since, the I/p can't be zero. Therefore the impulse response must be zero.
 i.e. $h[k] = 0$ for $k < 0$. Hence, the

expression for the causal system is $y[n] = \sum_{k=0}^{\infty} h[k] \cdot x[n-k]$.

$$x[n-k] = \sum_{k=0}^{\infty} x[k] \cdot h[n-k]. \text{ Hence, LTI system}$$

will be causal, $h[n] = 0$ for $n < 0$, which is the necessary and sufficient condition for an LTI system to be causal.

(ii) Stability:-

We know by convolution sum,

$$y[n] = x[n] \otimes h[n] = h[n] \otimes x[n]$$

Taking, $y[n] = h[n] \otimes x[n]$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k] \quad \text{--- (ii)}$$

Let x_p be bounded, i.e. $|x[n+k]| \leq M_x < \infty$
 Where 'M_x' is any finite integer.
 Now, The corresponding O/P is

$$y[n] = \left| \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]| \cdot |x[n-k]|$$

$$y[n] \leq \sum_{k=-\infty}^{\infty} |h[k]| \cdot M_x \quad \text{--- (ii)}$$

from eqn (ii), the system is stable if and only

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$$\therefore S_n = \sum_{k=-\infty}^{\infty} |h[k]| = \sum_{n=-\infty}^{\infty} h[n] < \infty$$

An LTI system will be stable if its impulse response is absolutely summable.

Q] $h[n] = \lambda^n u[n-2]$. Check the stability of the system having impulse response above.

So: Here, $h[n] = \lambda^n \cdot 1$; for $n-2 \geq 0$ i.e. $n \geq 2$
0 ; otherwise.

then for stability,

$$S_n = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$= \sum_{n=2}^{\infty} |\lambda|^n$$

$$= \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6 + \dots$$

$$= \lambda^2 [1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \dots]$$

$$= \lambda^2 \left[\frac{1}{1-\lambda} \right] < \infty$$

Hence, the system is stable.

$$Q.] h[n] = n \cdot \lambda^n \cdot u[n]$$

Here, $h[n] = n \cdot \lambda^n \cdot 1$; for $n \geq 0$
= 0 ; otherwise

then the stability,

$$S_n = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$= \sum_{n=0}^{\infty} |n \cdot \lambda^n|$$

Since, the n ranges 0 to ∞ .

$\therefore$ the sum will be infinite.

Hence, the system will be unstable.

Unit

Discrete time Fourier series (DTFS):

Consider a periodic discrete time sequence $x[n]$, having fundamental period (N) . Then, the DTFS is.

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 n k}$$

Where a_k is the Fourier series coefficient

$$\omega_0 = 2\pi/N = \frac{2\pi}{N}$$

Also, $a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\omega_0 n k}$; spectral

coefficient / Fourier series coefficient.

$$a_0 = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

Properties of DFS

(1) Periodicity:

If state that, if $x[n] \xleftrightarrow{\text{DTFS}} a_k$

Then $x[n+N] \xleftrightarrow{\text{DTFS}} a_k$

$$\omega_0 = 2\pi/N = \frac{2\pi}{N}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

proof.

We know,

$$\text{DTFS}\{x[n]\} = a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\omega_0 n k}$$

$$\therefore \text{DTFS}\{x[n+N]\} = \frac{1}{N} \sum_{n=0}^{N-1} x[n+N] e^{-j\omega_0 n k}$$

or, $\text{DTFS}\{x[n+N]\} = \frac{1}{N} \sum_{n=0}^{N-1} x[n+N] e^{-j\omega_0 n k}$

Let $n+N = p$; $n > 0 \therefore p < N$
 $\therefore n = p - N$; $n \rightarrow N-1 \therefore p \rightarrow 2N-1$

Diff. range

then, $\text{DTFS}\{x[n+N]\} = \frac{1}{N} \sum_{p=0}^{N-1} x[p] e^{j\omega_0 (p-N)k}$

$$= \left\{ \frac{1}{N} \sum_{p=0}^{N-1} x[p] e^{j\omega_0 p k} \right\} e^{j\omega_0 N k}$$

$$= a_k \cdot e^{j2\pi k} \quad [\because 2\pi = \omega_0 N]$$

$$= a_k \left[\because e^{j2\pi k} = \cos 2\pi k + j\sin 2\pi k \right]$$

$$= 1 + 0$$

$$= 1$$

(ii) Linearity properties:-
 If state that, if
 $x_1[n] \xrightarrow{\text{DFTSC}} a_k$
 & $x_2[n] \xrightarrow{\text{DFTSC}} b_k$

then,
 $\{ \alpha_1 x_1[n] + \alpha_2 x_2[n] \} \xrightarrow{\text{DFTSC}} \alpha_1 a_k + \alpha_2 b_k$
 i.e. linear sequence produces linear DFTSC.

Proof: We know,

$$\text{DFTSC } \{ x_1[n] \} = a_k = \frac{1}{N} \sum_{n=0}^{N-1} x_1[n] e^{-j\omega n k}$$

$$\text{DFTSC } \{ x_2[n] \} = b_k = \frac{1}{N} \sum_{n=0}^{N-1} x_2[n] e^{-j\omega n k}$$

Now, the linear combination of sequence.
 $x_1[n]$ and $x_2[n]$ is,

$$x_3[n] = \{ \alpha_1 x_1[n] + \alpha_2 x_2[n] \}$$

Now, the DFTSC of $x_3[n] = c_k$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_3[n] e^{-j\omega n k}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \{ \alpha_1 x_1[n] + \alpha_2 x_2[n] \} e^{-j\omega n k}$$

$$= \alpha_1 \cdot \frac{1}{N} \sum_{n=0}^{N-1} x_1[n] e^{-j\omega n k} + \alpha_2 \cdot \frac{1}{N} \sum_{n=0}^{N-1} x_2[n] e^{-j\omega n k}$$

$$= \alpha_1 a_k + \alpha_2 b_k$$

Hence proved, where, α_1 and α_2 are any arbitrary constant.

(iii) Time shifting:
 It states that if $x[n] \xrightarrow{\text{DFTSC}} a_k$

$$\text{then } x[n - n_0] \xrightarrow{\text{DFTSC}} a_k$$

Where,

n_0 is the constant time quantity.

Proof: We know,

$$\text{DFTSC } \{ x[n] \} = a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\omega n k}$$

$$\therefore \text{DFTSC } \{ x[n - n_0] \} = \frac{1}{N} \sum_{n=0}^{N-1} x[n - n_0] e^{-j\omega n k}$$

$$\text{Let } n - n_0 = p$$

$$n = p + n_0 \quad \text{then,}$$

$$\text{DFTSC } \{ x[n - n_0] \} = \frac{1}{N} \sum_{n=0}^{N-1} x[p] e^{-j\omega (p + n_0) k}$$

$$= \left(\frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot e^{-j\omega_0 n k} \right) \cdot e^{j\omega_0 n k}$$

$$\therefore \text{DIFS of } x(n-n_0) = a_k e^{-j\omega_0 n k}$$

$$x(n-n_0) \xrightarrow{\text{DIFS}} a_k e^{-j\omega_0 n k}$$

□

(iv) Differentiation property:-
It states that, if $x(n) \xrightarrow{\text{DIFS}} a_k$
then

$$\frac{d}{dn} x(n) \xrightarrow{\text{DIFS}} j\omega_0 k a_k$$

Proof: We know,
 $\text{DIFS of } x(n) = a_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\omega_0 n k}$

$$\text{Also, } x(n) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 n k} \quad \text{--- (i)}$$

Differentiating $x(n)$ w.r. to 'n' we get,

$$\frac{d}{dn} x(n) = \frac{d}{dn} \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 n k}$$

$$= \sum_{k=-\infty}^{\infty} a_k \cdot \frac{d}{dn} (e^{j\omega_0 n k})$$

$$= \sum_{k=-\infty}^{\infty} j\omega_0 k a_k e^{j\omega_0 n k}$$

$$= \sum_{k=-\infty}^{\infty} (j\omega_0 k a_k) e^{j\omega_0 n k} \quad \text{--- (ii)}$$

$$\left[\frac{d}{dn} x(n) \xrightarrow{\text{DIFS}} j\omega_0 k a_k \right] \quad \text{Hence proved.}$$

(v) Convolution property:-
It states that, $x_1(n) \xrightarrow{\text{DIFS}} a_k$
and $x_2(n) \xrightarrow{\text{DIFS}} b_k$
then,
 $x_1(n) \otimes x_2(n) \xrightarrow{\text{DIFS}} N a_k b_k$

It states that the multiplication in time domain sequence of the periodic sequence corresponds to multiplication in its spectral coefficient (fourier series coefficient).
Proof:

$$\text{We know, } \text{DIFS of } x_1(n) = a_k = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) e^{-j\omega_0 n k}$$

$$\text{DIFS of } x_2(n) = b_k = \frac{1}{N} \sum_{n=0}^{N-1} x_2(n) e^{-j\omega_0 n k}$$

Now,

$$\begin{aligned} \text{DFS of } \{x_1[n] \otimes x_2[n]\} &= \frac{1}{N} \sum_{n=0}^{N-1} \{x_1[n] \otimes x_2[n]\} e^{-j\omega n} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k] \right\} e^{-j\omega n} \end{aligned}$$

Now, interchanging the order of $\sum$ we get,

$$\text{DFS of } \{x_1[n] \otimes x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k] \left(\frac{1}{N} \sum_{n=0}^{N-1} x_2[n-k] e^{-j\omega n} \right)$$

By time shifting property, we have, (i)

$$\begin{aligned} \text{If } x[n] &\xrightarrow{\text{DFS}} a_k \\ \text{then } x[n-n_0] &\xrightarrow{\text{DFS}} a_k \cdot e^{-j\omega n_0} \end{aligned}$$

$$\begin{aligned} \text{i.e. } \text{DFS of } \{x_2[n-k]\} &= \frac{1}{N} \sum_{n=0}^{N-1} x_2[n-k] e^{-j\omega n} \\ &= b_k e^{-j\omega k} \end{aligned}$$

Hence, eq. (i) can be written as

$$\text{DFS of } \{x_1[n] \otimes x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k] b_k e^{-j\omega k}$$

$$= \left[\sum_{k=-\infty}^{\infty} x_1[k] e^{-j\omega k} \right] \cdot b_k$$

As,

$x_1[k]$ is a periodic sequence, hence the $\sum$ can be truncated as 0 to $N-1$

$$\begin{aligned} \therefore \text{DFS of } \{x_1[n] \otimes x_2[n]\} &= \left[\sum_{k=0}^{N-1} x_1[k] e^{-j\omega k} \right] \cdot b_k \\ &= N \left[\frac{1}{N} \sum_{k=0}^{N-1} x_1[k] e^{-j\omega k} \right] b_k \end{aligned}$$

$$\therefore \text{DFS of } \{x_1[n] \otimes x_2[n]\} = N \cdot a_k \cdot b_k$$

proved $\#$

Q.7 Find the spectral coefficients from the DFS expansion.

$$x[n] = 1 + \frac{1}{2} \sin \omega n + \frac{1}{\sqrt{2}} \sin \left(2\omega n + \frac{\pi}{4} \right)$$

We know,

$$x[n] = 1 + \frac{1}{2} \frac{\sin \omega n}{\frac{1}{2}} + \frac{1}{\sqrt{2}} \frac{\sin(2\omega n + \pi/4)}{\frac{1}{\sqrt{2}}}$$

$$x[n] = 1 + \frac{1}{2} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j} \right) + \frac{1}{\sqrt{2}} \left(\frac{e^{j(2\omega n + \pi/4)} - e^{-j(2\omega n + \pi/4)}}{2j} \right)$$

$$e^{j\pi/4} / e^{j\pi/4} = \cos \pi/4 + j \sin \pi/4$$

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$$= \frac{1}{2} + \frac{1}{4j} e^{j\omega n} - \frac{e^{-j\omega n}}{4j} + \frac{e^{j2\omega n} \cdot e^{j\pi/4}}{2\sqrt{2}j} - \frac{e^{j2\omega n} \cdot e^{j\pi/4}}{2\sqrt{2}j}$$

$$= \left(\frac{1-j}{4j} \right) e^{-j2\omega n} - \frac{e^{-j\omega n}}{4j} + 1 + \frac{e^{j\omega n}}{4j} + \left(\frac{1+j}{4j} \right) e^{j2\omega n}$$

Now,

Comparing eqⁿ (i) with standard Discrete time ∞ Fourier series.

$$x[n] = \sum_{k=-\infty}^{\infty} a_k \cdot e^{j\omega n k}$$

$$= a_{-2} e^{-j2\omega n} + a_{-1} e^{-j\omega n} + a_0 + a_1 e^{j\omega n} + a_2 e^{j2\omega n}$$

Equating eqⁿ (i) and (ii), we get,

$$a_{-2} = \frac{1-j}{4j}, \quad a_{-1} = -\frac{1}{4j}, \quad a_0 = 1$$

$$a_1 = \frac{1}{4j}, \quad a_2 = \frac{(1+j)}{4j}$$

$$\text{Again, } a_{-2} = \frac{1-j}{4j} \times \frac{j}{j} = -\frac{j}{4} - \frac{1}{4}, \quad a_{-1} = \frac{j}{4}$$

$$a_1 = -\frac{j}{4}, \quad a_2 = -\frac{j}{4} + \frac{1}{4}$$

DFT (Discrete time Fourier transform):

If the $x[n]$ is a non-periodic signal (sequence) then its corresponding Fourier transform is given by

$$X(e^{j\omega})$$

mathematically given by :-

$$x[n] \xrightarrow{\text{DFT}} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

It is the DFT of $x[n]$.

Similarly, IDFT (Inverse DFT) of $X(e^{j\omega})$ is

$x[n]$ given by

$$x[n] = \text{IDFT} \{ X(e^{j\omega}) \}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

IDFT of $x(e^{j\omega})$ eqⁿ (i) & (ii) are called Fourier transform pair.

Q. Find the DFT of $x[n] = e^{-an} \cdot u[n]$ for $a > 0$.

$\Rightarrow$ we know,

$$\text{By DFT, } X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{-an} \cdot u[n] \cdot e^{-j\omega n}$$

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

So, eqn (i) becomes,

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{\infty} e^{an} \cdot 1 \cdot e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} e^{-(a+j\omega)n} \\ &= 1 + e^{-(a+j\omega)} + e^{-2(a+j\omega)} + e^{-3(a+j\omega)} + \dots \\ &= \frac{1}{1 - e^{-(a+j\omega)}} \end{aligned}$$

Properties of DFT:-

(i) Linearity: It states that, if $x_1[n] \xrightarrow{\text{DFT}} X_1(e^{j\omega})$ & $x_2[n] \xrightarrow{\text{DFT}} X_2(e^{j\omega})$.

Then, $\alpha_1 \cdot x_1[n] + \alpha_2 \cdot x_2[n] \xrightarrow{\text{DFT}} \alpha_1 \cdot X_1(e^{j\omega}) + \alpha_2 \cdot X_2(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

→ It states that, the linear sequence of time domain corresponds linear in frequency domain

proof: We know,

$$X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1[n] \cdot e^{-j\omega n}$$

and

$$X_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2[n] \cdot e^{-j\omega n}$$

Now, the linear combination of the time domain sequence is $x_3[n] = \alpha_1 \cdot x_1[n] + \alpha_2 \cdot x_2[n]$. Then, the DFT of $x_3[n]$ is,

$$\begin{aligned} X_3(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_3[n] \cdot e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \{ \alpha_1 \cdot x_1[n] + \alpha_2 \cdot x_2[n] \} \cdot e^{-j\omega n} \\ &= \alpha_1 \cdot \sum_{n=-\infty}^{\infty} x_1[n] \cdot e^{-j\omega n} + \alpha_2 \cdot \sum_{n=-\infty}^{\infty} x_2[n] \cdot e^{-j\omega n} \\ &= \alpha_1 \cdot X_1(e^{j\omega}) + \alpha_2 \cdot X_2(e^{j\omega}) \end{aligned}$$

proved.

② Time shifting property:-

It states that if $x[n] \xrightarrow{\text{DFT}} X(e^{j\omega})$ then, $x[n-n_0] \xrightarrow{\text{DFT}} e^{-j\omega n_0} X(e^{j\omega})$

Proof: We know, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$

$$\text{Now, } \text{DFT}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0] \cdot e^{-j\omega n}$$

Let $n-n_0 = k$
 $n = k+n_0$ then,

$$\begin{aligned} \text{DFT}\{x[k]\} &= \sum_{k=-\infty}^{\infty} x[k] \cdot e^{-j\omega(k+n_0)} \\ &= \sum_{k=-\infty}^{\infty} x[k] \cdot e^{-j\omega k} \cdot e^{-j\omega n_0} \end{aligned}$$

$$= X(e^{j\omega}) \cdot e^{-j\omega n_0}$$

Hence,

$$x[n-n_0] \xrightarrow{\text{DFT}} e^{-j\omega n_0} X(e^{j\omega})$$

Proved

$$\Rightarrow -j\omega X(e^{j\omega})$$

3Q.7 Differentiation property:-

It states that if $x[n] \xrightarrow{\text{DFT}} X(e^{j\omega})$ then, $\frac{d}{dn} x[n] \xrightarrow{\text{DFT}} j\omega X(e^{j\omega})$

Proof:-

We know, by Inverse DFT,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega \quad \text{--- (i)}$$

Now, differentiating w.r. to 'n' we get,

$$\frac{d}{dn} x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot \left\{ \frac{d(e^{j\omega n})}{dn} \right\} d\omega$$

$$\text{or, } \frac{d}{dn} x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot j\omega e^{j\omega n} \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \{j\omega X(e^{j\omega})\} e^{j\omega n} \cdot d\omega \quad \text{--- (ii)}$$

from eq (i) and (ii),

$$\left[\frac{d}{dn} x[n] \xrightarrow{\text{DFT}} j\omega X(e^{j\omega}) \right]$$

Proved

(4) Convolution property :-

H states that,

$$x_1[n] \xleftrightarrow{\text{DFT}} X_1(e^{j\omega})$$

$$\text{and } x_2[n] \xleftrightarrow{\text{DFT}} X_2(e^{j\omega})$$

then,

$$x_1[n] \otimes x_2[n] \xleftrightarrow{\text{DFT}} X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$$

proof: we know by DFT

$$\begin{aligned} \text{DFT}\{x_1[n] \otimes x_2[n]\} &= \sum_{n=-\infty}^{\infty} (x_1[n] \otimes x_2[n]) e^{j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k] \right\} e^{j\omega n} \end{aligned}$$

Now,

inter changing the order of summation, we get

$$\text{DFT}\{x_1[n] \otimes x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k] \cdot \sum_{n=-\infty}^{\infty} x_2[n-k] e^{j\omega n}$$

we know by time shifting property,

$$\text{As, } x_2[n] \xleftrightarrow{\text{DFT}} X_2(e^{j\omega})$$

then,

$$x_2[n-n_0] \xleftrightarrow{\text{DFT}} e^{-j\omega n_0} X_2(e^{j\omega})$$

$$\text{i.e. } \sum_{n=-\infty}^{\infty} x_2[n-n_0] e^{j\omega n} = e^{-j\omega n_0} X_2(e^{j\omega})$$

Hence, eqn (i) can be written as;

$$\text{DFT}\{x_1[n] \otimes x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k] \cdot e^{-j\omega k} X_2(e^{j\omega})$$

$$= X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$$

proved.

Q. find DFT of $(\frac{1}{2})^n \cdot 4[n] \otimes (\frac{2}{5})^n \cdot 4[n]$.

$$\Rightarrow \text{let } x_1[n] = (\frac{1}{2})^n \cdot 4[n]$$

$$x_2[n] = (\frac{2}{5})^n \cdot 4[n]$$

$$\text{Now, } X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\frac{1}{2})^n \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{-j\omega}}{2} \right)^n$$

$$= 1 + \left(\frac{e^{-j\omega}}{2} \right) + \left(\frac{e^{-j\omega}}{2} \right)^2 + \dots$$

$$= \frac{1}{1 - \left(\frac{e^{-j\omega}}{2} \right)}$$

Again,

$$X_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2[n] \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{2e^{-j\omega}}{5}\right)^n$$

$$= 1 + \left(\frac{2e^{-j\omega}}{5}\right) + \left(\frac{2e^{-j\omega}}{5}\right)^2 + \dots$$

$$= \frac{1}{1 - \left(\frac{2e^{-j\omega}}{5}\right)}$$

Now,

$$x_1[n] \otimes x_2[n] = x_1(e^{j\omega}) \cdot x_2(e^{j\omega})$$

finally,

$$= \frac{1}{1 - \left(\frac{e^{-j\omega}}{2}\right)} \cdot \frac{1}{1 - \left(\frac{2e^{-j\omega}}{5}\right)}$$

Q.7 find DFT of $x[n] = 4\delta[n]$ and plot its magnitude & phase response.

By DFT, we have;

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} 1 e^{-j\omega n} \quad \left[\begin{array}{l} \text{for } n \geq 0 \\ 0, \text{ otherwise} \end{array} \right]$$

$$= \sum_{n=0}^{\infty} (e^{-j\omega})^n$$

$$= 1 + (e^{-j\omega}) + (e^{-j\omega})^2 + \dots$$

$$X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1 - \cos\omega + j\sin\omega} \quad \left[\begin{array}{l} e^{-j\omega} = \cos\omega + j\sin\omega \\ e^{j\omega} = \cos\omega - j\sin\omega \end{array} \right]$$

Now, magnitude,

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - (\cos\omega)^2 + \sin^2\omega}}$$

$$= \frac{1}{\sqrt{1 - 2\cos\omega + 1}}$$

$$= \frac{1}{\sqrt{2 - 2\cos\omega}}$$

phase,
 $\angle x(e^{j\omega}) = -\tan^{-1} \left(\frac{\sin \omega}{1 - \cos \omega} \right)$

Now,

ω	$ x(e^{j\omega}) $	$\angle x(e^{j\omega})$
0	∞	0
$+\pi/4$		
$-\pi/4$		
$+\pi/2$		
$-\pi/2$		

Lab:-
 # Even and odd signals:-

for even signal:

$x[n] = x[-n]$; for discrete time domain
 $x(t) = x(-t)$; for continuous time domain.

for odd signal:-

$x[n] = -x[-n]$; for DTD
 $x(t) = -x(-t)$; for CTD.

Decomposition of signal in its even and odd parts:-

Consider a discrete time domain sequence $x[n]$ which can be decomposed in to its even parts and odd parts as $x_e[n]$ and $x_o[n]$ respectively.
 then, $x[n]$ can be written as,

$$x[n] = x_e[n] + x_o[n] \quad \text{--- (i)}$$

for, replacing 'n' by '-n', we get.
 $x[-n] = x_e[-n] + x_o[-n]$

$$\text{or, } x[-n] = x_e[n] - x_o[n] \quad \text{--- (ii)}$$

Now, solving eqn (i) & (ii), we get,

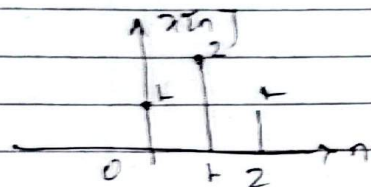
$$\therefore x_e[n] = \frac{x[n] + x[-n]}{2}$$

$$\therefore x_o[n] = \frac{x[n] - x[-n]}{2}$$

if continuous time domain:-

$$x_e(t) = \frac{x(t) + x(-t)}{2} \quad , \quad x_o(t) = \frac{x(t) - x(-t)}{2}$$

Q.7] sketch $x[n]$ and $x_e[n]$ for $x[n] = \{1, 2, 1\}$



We know,

$$x_e[n] = \frac{x[n] + x[-n]}{2}$$

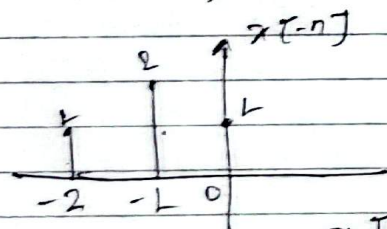
Where,

$$x[-n]$$

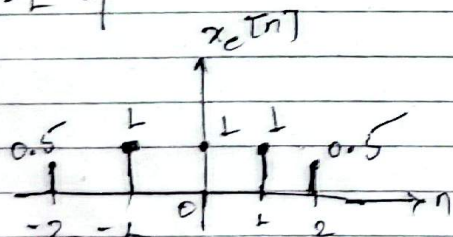
When $n=0$, $x[-n] = x[0] = 1$.

When $n=1$, $x[-n] = x[-1] = 2$.

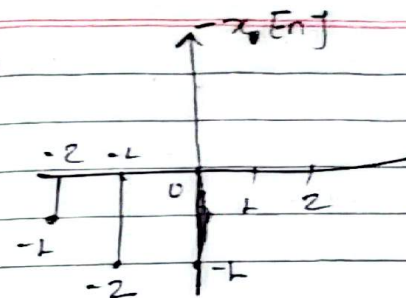
When $n=2$, $x[-n] = x[-2] = 1$.



Now,

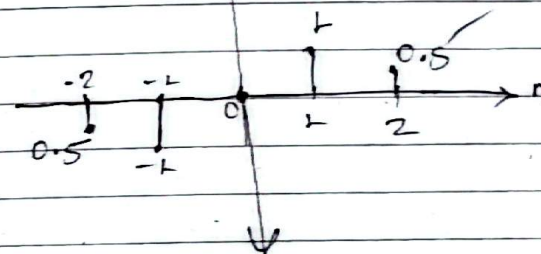


Again,



Now,

$$x_e[n] = \frac{x[n] + \{-x[-n]\}}{2}$$



Code:

```
n = -5:5;
x = [1, 2, 1];
y = flipr(x);
even = 0.5 * (x + y);
stem(even)
```

Command:

$x = [1, 2, 1]$

$y = \text{fliplr}(x)$

$x_2 = \dots, h = \dots$

$y = \text{conv}(x, h)$

Code

```
n = -5:5;
x = [ ];
h = [ ];
y = conv(x, h);
stem(y);
```


Dirichlet's Condition for the existence of Fourier series / Fourier transform :-

(i) Fourier series :-

- The periodic sequence $x[n]$ must be absolutely summable. i.e. $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

Note: If periodic sequence is continuous time domain $x(t)$ then integrate them.

- The sequence $x[n]$ must have finite number of minima and maxima.
- The sequence $x[n]$ must have finite number of discontinuities.

(ii) Fourier transform :-

- The non-periodic sequence $x(t)$ must be absolutely summable.
- The sequence $x(t)$ must have finite number of minima & maxima.
- It must have finite no. of discontinuities.

20 marks!

z-transform :-

Consider discrete sequence $x[n]$ whose z-transform is $x[n] \xrightarrow{z} X[z]$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} \quad \text{--- (i)}$$

$$= \underbrace{\sum_{n=-\infty}^{-1} x[n] \cdot z^{-n}}_{\text{Non-Causal}} + \underbrace{\sum_{n=0}^{\infty} x[n] \cdot z^{-n}}_{\text{Causal}} \quad \text{--- (ii)}$$

From Eqⁿ (i) :-
 → The z-transform $\sum_{n=-\infty}^{\infty}$ corresponds to Non Causal z-transform.

→ The z-transform from $n=0$ to ∞ corresponds to Causal z-transform.

→ The z-transform having both side of n is called two sided z-transform.

→ For Causal:
 $X[z] = \sum_{n=0}^{\infty} x[n] \cdot z^{-n}$

→ for Non-Causal:
 $X[z] = \sum_{n=-\infty}^{-1} x[n] \cdot z^{-n}$

ROC (Region of Convergence) :-
 All the values of z in which $X[z]$ is finite. is called it's ROC.

Q.7 Find the z-transform of $u[n]$.

We know,
 $u[n] = 1$; for $n \geq 0$
 0 ; otherwise.

Then, z-transform $u[n]$ is,

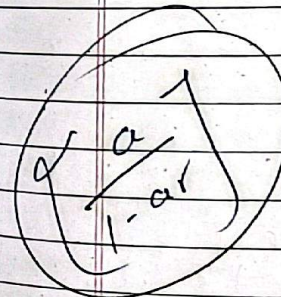
$$Z\{u[n]\} = \sum_{n=-\infty}^{\infty} u[n] \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= 1 + (z^{-1}) + (z^{-1})^2 + \dots$$

$$= \frac{1}{1 - z^{-1}} \quad \#$$



for ROC:

$$|z| < 1;$$

$$\text{i.e. } \frac{1}{|z|} < 1$$

i.e. $|z| > 1$; Required ROC.

Q7] Find the z-transform of $u[n-1]$

We know,

$$u[n-1] = 1; \text{ for } -n-1 \geq 0 \text{ i.e. } n \leq -1$$

0; otherwise i.e. $n \geq 0$

Now,

$$Z\{u[n-1]\} = \sum_{n=-\infty}^{\infty} u[n-1] \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{-1} 1 \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (z^{-1})^n$$

$$= \dots + z^4 + z^3 + z^2 + z$$

$$= z + z^2 + z^3 + z^4 + \dots$$

$$= z(1 + z + z^2 + z^3 + \dots)$$

$$= \frac{z}{1-z} \quad \text{--- (4)}$$

$$= \frac{-1}{1-z^{-1}}, \text{ for } |z| < 1.$$

--- main formula.

Q7] Find the z-transform of $x[n] = \sin \omega_0 n \cdot u[n]$

$$= \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} u[n]$$

$$= \frac{e^{j\omega_0 n}}{2j} - \frac{e^{-j\omega_0 n}}{2j} \text{ for } n \geq 0$$

0; otherwise

Now,

z-transform, we get,

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega_0 n}}{2j} z^{-n} - \sum_{n=0}^{\infty} \frac{e^{-j\omega_0 n}}{2j} z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n - \frac{1}{2j} \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^n$$

$$= \frac{1}{2j} \frac{1}{1 - e^{j\omega_0} z^{-1}} - \frac{1}{2j} \frac{1}{1 - e^{-j\omega_0} z^{-1}} \quad \text{--- (5)}$$

for loc, 1st part = $|e^{j\omega_0} z^{-1}| < 1$

$$\text{or, } \left| \frac{e^{j\omega_0}}{z} \right| < 1$$

$$\text{i.e. } |z| > |e^{j\omega_0}| = |\cos\omega_0 + j\sin\omega_0|$$

$$= \sqrt{\cos^2\omega_0 + \sin^2\omega_0}$$

$$= 1$$

$$\text{i.e. } |z| > 1.$$

2nd part, $|e^{-j\omega_0} z^{-1}| < 1$

$$\left| \frac{e^{-j\omega_0}}{|z|} \right| < 1$$

$$\text{i.e. } |z| > 1.$$

solving eqn (1), we get,

$$X(z) = \frac{1}{zj} \left(\frac{1 - e^{-j\omega_0} z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})} - \frac{1 + e^{j\omega_0} z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})} \right)$$

$$= \left(\frac{e^{j\omega_0} - e^{-j\omega_0}}{zj} \right) z^{-1}$$

$$(1 - e^{-j\omega_0} z^{-1}) (1 - e^{j\omega_0} z^{-1} + z^{-2})$$

$$= \frac{(\sin\omega_0) z^{-1}}{1 - 2z^{-1} \left(\frac{e^{j\omega_0} + e^{-j\omega_0}}{2} \right) + z^{-2}}$$

$$= \frac{(\sin\omega_0) z^{-1}}{1 - 2z^{-1} \cos\omega_0 + z^{-2}}, \text{ ROC } |z| > 1$$

Properties of z-transform :-

①

Linearity :

i.e. state that if $x_1[n] \xleftrightarrow{Z} X_1(z)$
and $x_2[n] \xleftrightarrow{Z} X_2(z)$ then,

$$\alpha_1 \cdot x_1[n] + \alpha_2 \cdot x_2[n] \xleftrightarrow{Z} \alpha_1 \cdot X_1(z) + \alpha_2 \cdot X_2(z)$$

Where, α_1 and α_2 are arbitrary constant.

i.e. linear in time domain corresponds to linear z-domain.

proof:

We know

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] \cdot z^{-n}$$

&

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n] \cdot z^{-n}$$

Then, the linear combination of time domain sequence is,

$$x[n] = \alpha_1 \cdot x_1[n] + \alpha_2 \cdot x_2[n]$$

Now, then z-transform of $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \{ \alpha_1 \cdot x_1[n] + \alpha_2 \cdot x_2[n] \} z^{-n}$$

$$= \alpha_1 \sum_{n=-\infty}^{\infty} x_1[n] \cdot z^{-n} + \alpha_2 \sum_{n=-\infty}^{\infty} x_2[n] \cdot z^{-n}$$

$$= \alpha_1 \cdot X_1(z) + \alpha_2 \cdot X_2(z)$$

Hence proved.

②

Time shifting property :-
i.e. state that if $x[n] \xleftrightarrow{Z} X(z)$
then $x[n-n_0] \xleftrightarrow{Z} z^{-n_0} X(z)$
Where ' n_0 ' is constant time quantity.

proof:

We know,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} = Z \{ x[n] \}$$

$\therefore$

$$Z \{ x[n-n_0] \} = \sum_{n=-\infty}^{\infty} x[n-n_0] \cdot z^{-n} \quad (1)$$

Let $n - n_0 = m$

$$n = m + n_0 ; \text{ As } n \rightarrow -\infty ; m \rightarrow -\infty$$

$$n \rightarrow \infty ; m \rightarrow \infty$$

then,

$$Z \{ x[m] \} = \sum_{m=-\infty}^{\infty} x[m] \cdot z^{-(m+n_0)}$$

$$= z^{-n_0} \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= z^{-n_0} X(z)$$

$$\therefore x[n-n_0] \xrightarrow{z} z^{-n_0} X(z) \text{ proved.}$$

(iii) Scaling properties:-
 it state that if $x[n] \xrightarrow{z} X(z)$
 then $a^n x[n] \xrightarrow{z} X(a^{-1}z)$
 where a is any constant.

Proof: we know, $\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$
 then,

$$\begin{aligned} \mathcal{Z}\{a^n \cdot x[n]\} &= \sum_{n=-\infty}^{\infty} a^n \cdot x[n] \cdot z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n] \cdot (a^{-1}z)^{-n} \\ &= X(a^{-1}z) \end{aligned}$$

$$\therefore \mathcal{Z}\{a^n \cdot x[n]\} = X(a^{-1}z) \text{ proved.}$$

Note: Transfer function $H(z) = \frac{Y(z)}{X(z)}$

Q. find the transfer function $H(z)$ from the given differential eqn.

$$y[n] = y[n-1] - \frac{1}{2} y[n-2] + x[n] + \frac{1}{2} x[n-1]$$

$\Rightarrow$ Taking z -transform, we get,

$$\begin{aligned} \mathcal{Z}\{y[n]\} &= \mathcal{Z}\{y[n-1]\} - \frac{1}{2} \mathcal{Z}\{y[n-2]\} + \mathcal{Z}\{x[n]\} \\ &\quad + \frac{1}{2} \mathcal{Z}\{x[n-1]\} \end{aligned}$$

$$\begin{aligned} \text{or, } Y(z) &= z^{-1} Y(z) - \frac{1}{2} z^{-2} Y(z) + X(z) \\ &\quad + \frac{1}{2} z^{-1} X(z) \end{aligned}$$

$$\begin{aligned} \text{or, } Y(z) - z^{-1} Y(z) + \frac{1}{2} z^{-2} Y(z) &= X(z) + \frac{1}{2} z^{-1} X(z) \\ Y(z) \left(1 - z^{-1} + \frac{1}{2} z^{-2}\right) &= X(z) \left(1 + \frac{1}{2} z^{-1}\right) \\ \frac{Y(z)}{X(z)} &= \frac{\left(1 + \frac{1}{2} z^{-1}\right)}{\left(1 - z^{-1} + \frac{1}{2} z^{-2}\right)} \end{aligned}$$

$$\therefore H(z) = \frac{1 + \frac{1}{2} z^{-1}}{1 - z^{-1} + \frac{1}{2} z^{-2}}$$

Q7 find the Z-transform of $x[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$

⇒ We know,
 $u[n] \xrightarrow{Z} U(z) = \frac{1}{1 - z^{-1}}$

By scaling property,

$$x[n] \xrightarrow{Z} X(z)$$

then,

$$a^{-1}x[n] \xrightarrow{Z} X(a^{-1}z)$$

Now,

$$Z\{x[n]\} = Z\left\{\left(\frac{1}{2}\right)^n \cdot u[n]\right\} = \frac{1}{1 - \left(\frac{1}{2}\right)^{-1} z^{-1}}$$

$$= \left(\frac{1}{1 - \frac{1}{2} z^{-1}} \right) \quad \#$$

Q7 find the $x[n]$ by using time shifting and scaling property.

$$X(z) = \frac{1 + \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

⇒ Soln:

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{\frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$= X_1(z) + \frac{1}{2} z^{-1} X_1(z)$$

Where, $X_1(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$

∴ Taking inverse Z-transform by scaling property,

$$x_1[n] = \left(\frac{1}{2}\right)^n \cdot u[n] \quad \text{--- (1)}$$

Also, $x_1[n-1] \xrightarrow{Z} z^{-1} X_1(z)$

from eqn (1), also $x_1[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$

$$x_1[n-1] = \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$\therefore x[n] = x_1[n] + \frac{1}{2} x_1[n-1]$$

$$= \left(\frac{1}{2}\right)^n \cdot u[n] + \frac{1}{2} \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$= \left(\frac{1}{2}\right)^n [u[n] + u[n-1]] \quad \#$$

Differentiation property:-

It states that if $x(n) \xrightarrow{Z} X(z)$
then $n \cdot x(n) \xrightarrow{Z} -z \frac{d}{dz} X(z)$

Proof: We know by definition Z-transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad (i)$$

Differentiating w.r. to 'z' we get,

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot (-n) \cdot z^{-n-1}$$

$$= - \sum_{n=-\infty}^{\infty} n [x(n)] \cdot z^{-n}$$

$$= -z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n x(n) z^{-n} \quad (ii)$$

from eqⁿ (ii), it is clear that the Z-transform
of $n \cdot x(n) \xrightarrow{Z} -z \frac{d}{dz} X(z)$
proved

check again,

8] find $X(z)$ for

$$x(n) = n \cdot u(n)$$

We know,

$$Z\{u(n)\} = V(z) = \frac{1}{1-z^{-1}}$$

Now, By Differentiation property, we get,

$$x(n) = n \cdot u(n) \xrightarrow{Z} X(z) = -z \frac{d}{dz} V(z) \quad (i)$$

$$\frac{d}{dz} V(z) = \frac{d}{dz} \left(\frac{1}{1-z^{-1}} \right)$$

$$= \frac{d}{dz} \left(\frac{1}{1-z^{-1}} \right)$$

$$= \left[\frac{1-z^{-1}(0) - 1 \cdot (-1) \cdot z^{-2}}{(1-z^{-1})^2} \right]$$

$$= \frac{z^{-1}}{(1-z^{-1})^2}$$

Q7. $x(n) = n \left(\frac{1}{2}\right)^n u(n)$. find $X(z)$.

⇒ We know,

$$x(n) \xleftrightarrow{Z} X(z)$$

$$n \cdot \left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{Z} X(z) = n \cdot \left(\frac{1}{2}\right)^n \cdot \frac{d}{dz} \left\{ \frac{1}{1 - \frac{1}{2}z^{-1}} \right\}$$

Let $x(n) = n \cdot x_1(n)$.

$$\therefore X_1(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Now, $x(z) = n \cdot x_1(n) \xleftrightarrow{Z} -z \cdot \frac{d}{dz} \{X_1(z)\}$

$$X(z) = -z \frac{d}{dz} \left(\frac{1}{1 - \frac{1}{2}z^{-1}} \right)$$

$$= -z \left(\frac{(1 - \frac{1}{2}z^{-1}) \cdot 0 - 1 \cdot (0 - \frac{1}{2}z^{-2}(-1))}{(1 - \frac{1}{2}z^{-1})^2} \right)$$

$$= \frac{\frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})^2}$$

Q7 find $x(n)$ from $X(z) = \log(1 + az^{-1})$; for $|a| < 1$.

⇒ We know,

By Diff. property, $n \cdot x(n) \xleftrightarrow{Z} z \frac{d}{dz} X(z)$

$$X(z) = \frac{d}{dz} \{x(n)\} = \frac{d}{dz} (\log(1 + az^{-1})) \cdot (-z)$$

$$= \frac{1}{1 + az^{-1}} \cdot a \cdot (-1) \cdot z^{-2} \cdot (-z)$$

$$= \frac{-a z^{-2} (-z)}{1 + az^{-1}}$$

$$= \frac{a z^{-1}}{1 + az^{-1}} \quad \text{①}$$

We have,

$$y(n) \xleftrightarrow{Z} Y(z) = \frac{1}{1 - z^{-1}}$$

$$(-a)^n \cdot y(n) \xleftrightarrow{Z} Y(z) = \frac{1}{1 - (-a)z^{-1}}$$

Also, By time shifting property,

$$x(n) \xleftrightarrow{Z} X(z)$$

$$x(n-1] \xleftrightarrow{Z} z^{-1} X(z)$$

eg. $(-a)^n u(n) \xleftrightarrow{Z} \frac{1}{1+az^{-1}}$

$$\therefore (-a)^{n-1} u(n-1) \xleftrightarrow{Z} \frac{-z^{-1}}{1+az^{-1}} \quad \text{--- (ii)}$$

Hence, from eqⁿ (i) & (ii), we get,

$$n \cdot x(n) = a(-a)^{n-1} u(n-1)$$

$$\therefore x(n) = (-1)^{n-1} \frac{a^n}{n} u(n-1)$$

2.m.f

Convolution property :-
It state that if $x_1(n) \xleftrightarrow{Z} X_1(z)$
and $x_2(n) \xleftrightarrow{Z} X_2(z)$ then $x_1(n) \otimes x_2(n) \xleftrightarrow{Z} X_1(z) \cdot X_2(z)$. It means that

Convolution in time domain corresponds to multiplication in Z-domain.

proof:- We know, ∞
 $X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot z^{-n}$
and $X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) \cdot z^{-n}$

Then,
$$\begin{aligned} Z\{x_1(n) \otimes x_2(n)\} &= \sum_{n=-\infty}^{\infty} \{x_1(n) \otimes x_2(n)\} \cdot z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} \{x_1(k) \cdot x_2(n-k)\} \right\} \cdot z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1(k) \left\{ \sum_{n=-\infty}^{\infty} x_2(n-k) \cdot z^{-n} \right\} \quad \text{--- (i)} \end{aligned}$$

Now, By time shifting property, we know,
 $x(n) \xleftrightarrow{Z} X(z)$

$$\therefore x(n-k] \xleftrightarrow{Z} z^{-k} X(z) = \sum_{n=-\infty}^{\infty} x(n-k) \cdot z^{-n}$$

from eqn ① becomes,

$$Z\{x_1(n) \otimes x_2(n)\} = \sum_{k=-\infty}^{\infty} x_1(k) \cdot z^{-k} \cdot X_2(z)$$

$$= X_1(z) \cdot X_2(z)$$

Hence proved.

Q7] find the z-transform of $(\frac{1}{2})^n \cdot u(n) \otimes (\frac{1}{4})^n u(n)$

⇒ let.

$$x_1(n) = (\frac{1}{2})^n u(n)$$

$$x_2(n) = (\frac{1}{4})^n u(n)$$

$$Z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} (\frac{1}{2})^n \cdot u(n) \cdot z^{-n}$$

$$X_1(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

similarly,

$$X_2(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

Now, By convolution property,

$$\{(\frac{1}{2})^n u(n) \otimes (\frac{1}{4})^n u(n)\} = X_1(z) \cdot X_2(z)$$

$$= \left(\frac{1}{1 - \frac{1}{2}z^{-1}} \right) \cdot \left(\frac{1}{1 - \frac{1}{4}z^{-1}} \right)$$

$$= \left(\frac{1}{z - \frac{1}{2}} \right) \cdot \frac{1}{z - \frac{1}{4}}$$

$$= \left(\frac{z}{z - \frac{1}{2}} \right) \cdot \left(\frac{z}{z - \frac{1}{4}} \right)$$

$$= \frac{z^2}{(z - \frac{1}{2}) \cdot (z - \frac{1}{4})}$$

$$= \frac{z^2}{z^2 - \frac{1}{4}z - \frac{1}{2}z + \frac{1}{8}}$$

$$= \left(\frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}} \right) \quad \#$$

Inverse z-transform :-

- (1) long division method
- (2) partial fraction

(1) long division method :-

find $x(n)$ for $X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$

(i) If $x(n)$ is causal

$$\begin{array}{r}
 1 + 1.5z^{-1} + 1.75z^{-2} \\
 1 - 1.5z^{-1} + 0.5z^{-2} \overline{) \phantom{1.75z^{-2}}} \\
 \underline{-x - 1.5z^{-1} + 0.5z^{-2}} \\
 1.5z^{-1} - 0.5z^{-2} \\
 \underline{-1.5z^{-1} + 2.25z^{-2} + 0.75z^{-3}} \\
 1.75z^{-2} - 0.75z^{-3} \\
 \underline{1.75z^{-2}} \\
 1.75z^{-2}
 \end{array}$$

$$X(z) = 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + \dots \quad (1)$$

We know,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \dots$$

Comparing eqn (i) and (ii) we get, — (ii)

$$x(0) = 1$$

$$x(1) = 1.5$$

$$x(2) = 1.75$$

$$x(3) = 1.875$$

So,

$$x(n) = \{1, 1.5, 1.75, 1.875, \dots\}$$

(ii) If $x(n)$ is anticausal :-

$$\begin{array}{r}
 2z^2 + 6z^3 + 12z^4 \\
 0.5z^{-2} - 1.5z^{-1} + 1 \overline{) } \\
 \underline{-x - 3z + 2z^2} \\
 3z - 2z^2 \\
 \underline{-3z + 8z^2 + 6z^3} \\
 6z^2 - 6z^3 \\
 \underline{-6z^2 + 18z^3 + 12z^4} \\
 12z^3 - 12z^4
 \end{array}$$

$$X(z) = 2z^2 + 6z^3 + 12z^4 + \dots$$

$$= \dots + 12z^4 + 6z^3 + 2z^2 + \dots \quad (i)$$

We know for anticausal $x(n)$

$$X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n}$$

$$= \dots + x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + \dots \quad (ii)$$

Comparing (i) and (ii),

$$x(-4) = 12$$

$$x(-3) = 6$$

$$x(-2) = 2$$

$$x(-1) = 0$$

$$x(0) = 0$$

$$\therefore x(n) = \{ \dots, 12, 6, 2, 0, 0 \}$$

Inverse z-transform by partial fraction method

$$H(z) = C + h_1(z) + h_2(z) + h_3(z) + \dots + h_N(z)$$

Where, $h_1(z), h_2(z), h_3(z), \dots, h_N(z)$ all are applied with inverse z-transform to obtain

$h(n)$ as;

$$h(n) = C + h_1(n) + h_2(n) + h_3(n) + \dots + h_N(n)$$

Q.) find inverse z-transform of

$$H(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$= \frac{1}{1 - \frac{1.5}{z} + \frac{0.5}{z^2}}$$

$$= \frac{z^2}{z^2 - 1.5z + 0.5}$$

$$= \frac{z^2}{z^2}$$

$$\frac{H(z)}{z} = \frac{(z-1)(z-0.5)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}$$

↓ Check something wrong.

By partial fraction,

$$z = A(z-0.5) + B(z-1)$$

$$A = 2$$

$$B = \cancel{0.5} - 1$$

$$\frac{H(z)}{z} = \frac{2}{(z-1)} - \frac{\cancel{0.5}}{z-0.5}$$

$$\text{or, } H(z) = \frac{2z}{(z-1)} - \frac{\cancel{0.5}z}{(z-0.5)}$$

$$= \left(\frac{2}{1-z^{-1}} \right) - \cancel{0.5} \left(\frac{1}{1-0.5z^{-1}} \right) \rightarrow \text{Ⓜ}$$

Now taking inverse z-transform eqⁿ Ⓜ

$$h(n) = 2 \cdot u(n) - (0.5)^n \cdot u(n)$$

Q] Find $h(n)$ from $H(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$

When

- a) ROC : $|z| > 1$
 b) ROC : $|z| < 0.5$
 c) ROC : $0.5 < |z| < 1$

$$\text{Here, } H(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$= \frac{z^2}{z^2-1.5z+0.5}$$

$$\frac{H(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}$$

$$A = 2$$

$$B = -1$$

$$\frac{H(z)}{z} = \frac{2}{(z-1)} - \frac{1}{(z-0.5)}$$

$$H(z) = \frac{2z}{(z-1)} - \frac{z}{(z-0.5)}$$

$$= 2 \left(\frac{1}{1-z^{-1}} \right) - \left(\frac{1}{1-0.5z^{-1}} \right) \rightarrow \text{Ⓜ}$$

1st part 2nd part

Note: $\sum_{k=L}^N \frac{A_k}{1-p_k z^{-1}} \xleftrightarrow{z^{-1}} \sum_{k=L}^N A_k (p_k)^n u(n-L)$ for causal signal

$-\sum_{k=L}^N A_k (p_k)^n u(n-L)$ for Non-causal

$$|z^{-1}| < 1$$

$$i.e. |z| > 1$$

$$|0.5z^{-1}| < 1$$

$$i.e. |z| > 0.5$$

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a) When ROC: $|z| > 1$, In this case both the part are causal.

Taking inverse z-transform,

$$h(n) = 2u(n) - (0.5)^n u(n)$$

b) When ROC: $|z| < 0.5$, The both case are non-causal,

$$h(n) = -2[u(-n-1)] - (0.5)^n u(-n-1)$$

$$= [-2 + (0.5)^n] u(-n-1)$$

c) When ROC: $(0.5 < |z| < 1)$

$|z| > 0.5$ $|z| < 1$, In this case the 1st part is non-causal and 2nd part is causal.

so, Taking z^{-1} transform,

$$h(n) = -2u(-n-1) + (0.5)^n u(n)$$

Q.7 Determine $h(n)$ from the LCCD eqn.

$$3y[n] = 4y[n-1] - y[n-2] + x[n] + \frac{1}{4}(x[n-1])$$

Now, Taking z-transform on both side,

~~$$3y(z) = 4z^{-1}y(z) - z^{-2}y(z) + X(z) + \frac{1}{4}z^{-1}X(z)$$~~

or,

$$3y(z) = 4z^{-1}y(z) - z^{-2}y(z) + X(z) + \frac{1}{4}z^{-1}X(z)$$

$$3y(z) - 4z^{-1}y(z) + z^{-2}y(z) = X(z) + \frac{1}{4}z^{-1}X(z)$$

$$y(z)(3 - 4z^{-1} + z^{-2}) = X(z)(1 + \frac{1}{4}z^{-1})$$

$$\frac{y(z)}{X(z)} = \frac{1 + \frac{1}{4}z^{-1}}{(z^{-2} - 4z^{-1} + 3)}$$

$$H(z) = \frac{1 + \frac{1}{4}z^{-1}}{z^{-2} - 4z^{-1} + 3}$$

Again,

$$H(z) = \frac{1 + \frac{1}{4z}}{\frac{1}{z^2} - \frac{3}{z} + 3}$$

$$= \frac{(4z+1)}{4z} \cdot \frac{z^2}{1 - 4z + 12z^2}$$

$$= \frac{z^2(4z+1)}{4z - 16z^2 + 12z^3}$$

$$= \frac{z(z + \frac{1}{4})}{3z^2 - 4z + 1}$$

$$\frac{H(z)}{z} = \frac{z + \frac{1}{4}}{(3z-1)(z-1)} = \frac{A}{(3z-1)} + \frac{B}{(z-1)}$$

By partial fraction,

$$z + \frac{1}{4} = A(z-1) + B(3z-1)$$

$$\therefore A = 5/8$$

$$B = -7/8$$

$$H(z) = \frac{5/8 \cdot z}{(3z-1)} - \frac{7/8 \cdot z}{(z-1)}$$

$$= \frac{5/8}{3(1 - \frac{1}{3}z^{-1})} - \frac{7/8}{1(1 - z^{-1})}$$

$$= \frac{5}{24} \left(\frac{1}{1 - \frac{1}{3}z^{-1}} \right) - \frac{7}{8} \left(\frac{1}{1 - z^{-1}} \right)$$

ROC: 1st part $|\frac{1}{3}z^{-1}| < 1$ / 2nd part $|z^{-1}| < 1$
i.e. $|z| > \frac{1}{3}$

Now, taking z^{-1} transform,

$$h(n) = \frac{5}{24} \left(\frac{1}{3} \right)^n u(n) - \frac{7}{8} u(n)$$

Q.7] In the above eqn find $h(n)$ and specify ROC.

- a) for causal signal
- b) for non-causal signal
- c) for stable signal

Now, solving we get,

$$h(n) = \frac{5}{24} \left(\frac{1}{3}\right)^n u(n) - \frac{7}{8} u(n)$$

(a) When the sequence is causal,
for: $|z| > 1$

By taking z^{-1} transform,

$$h(n) = -\frac{7}{24} \left(\frac{1}{3}\right)^n u(n) + \frac{5}{8} u(n)$$

(b) When the system is anticausal/ (Non causal) $\frac{u(n)}{8}$
for: $|z| < \frac{1}{3}$

Take z^{-1} transform,

$$h(n) = -\frac{7}{24} \left(\frac{1}{3}\right)^n u(-n-1) - \frac{5}{8} u(-n-1)$$

$$= \frac{7}{24} \left(\frac{1}{3}\right)^n u(n-1) - \frac{5}{8} u(n-1)$$

(c) When the system is stable,
for the system to be stable the for
must be bounded,

$$\text{i.e. } \frac{1}{3} < |z| < 1$$

$$\text{i.e. } |z| > \frac{1}{3}, |z| < 1$$

the this is the 1st part of causal &
2nd part $|z| > 1$ is not causal,

By taking z^{-1} transform,

$$h(n) = \frac{7}{24} \left(\frac{1}{3}\right)^n u(n) + \frac{5}{8} (-u(n-1))$$

$$= \frac{7}{24} \left(\frac{1}{3}\right)^n u(n) - \frac{5}{8} u(n-1)$$

find the unit step response of an LTI system
Describe by the difference eqn

$$3y(n) = 4y(n-1) - y(n-2) + 2(n) + \frac{1}{4} x(n-1)$$

find $H(z)$ find, $x(n) = \delta(n)$ is unit step $x(n) = u(n)$

$$X(z) = \left(\frac{1}{z-1}\right) = \left(\frac{z}{z-1}\right) = \begin{pmatrix} z \\ 1 \end{pmatrix}$$

$$y(n) = ?$$

$$Y(z) = H(z) \cdot X(z)$$

$$= \text{do it}$$

$$\rightarrow \frac{y(z)}{z} = \frac{A}{(-)} + \frac{B}{(-)} + \frac{C}{(-)^2}$$

Q.7 $H(z) = \frac{(1+z^{-1})^2}{(1-0.5z^{-1})(1+0.75z^{-1})}$

find difference equation satisfying the i/p and o/p of the given system $H(z)$.

$$\Rightarrow H(z) = \frac{(1+z^{-1})^2}{(1+0.75z^{-1})(1-0.5z^{-1})}$$

$$\frac{y(z)}{x(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 + 0.75z^{-1} - 0.5z^{-1} - 0.375z^{-2}}$$

$$= \frac{z^2 + 2z^{-1} + 1}{1 + 0.25z^{-1} - 0.375z^{-2}}$$

$$y(z) + 0.25z^{-1}y(z) - 0.375z^{-2}y(z) = z^2x(z)$$

By taking z^{-1} transform we get,

$$y(n) + 0.25y(n-1) - 0.375y(n-2) = x(n-2) + x(n-1) + x(n)$$

Again, find $h(n)$:-

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + 0.25z^{-1} - 0.375z^{-2}} = 1 + \frac{2}{z} + \frac{1}{z^2}$$

$$H(z) = \frac{(z^2 + 2z + 1)}{(z^2 + 0.25z - 0.375)} = z \left(z + 2 + \frac{1}{z} \right) \frac{1}{(1 + 0.75/z)(1 - 0.5/z)}$$

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{(1 + 0.75z^{-1})(1 - 0.25z^{-1})}$$

$$= \frac{z^2 + 2z + 1}{z(z - 0.75)(z + 0.25)}$$

$$= \frac{1 + \frac{2}{z} + \frac{1}{z^2}}{(1 - 0.25/z)(1 + 0.75/z)}$$

$$= \frac{(z^2 + 2z + 1)}{z^2(z - 0.25)(z + 0.75)}$$

$$= \frac{(z^2 + 2z + 1)}{z(z - 0.25)(z + 0.75)}$$

$$= \frac{z^2 + 2z + 1}{(z - 0.25)(z + 0.75)}$$

$$H(z) = \frac{(z + 2 + \frac{1}{z})}{(z - 0.25)(z + 0.75)}$$

$$= \frac{A}{(z - 0.25)} + \frac{B}{(z + 0.75)}$$

$$(z + 2 + \frac{1}{z}) = A(z + 0.75) + B(z - 0.25)$$

Solving, $z = 0.25$, then,

$$A = 6.25$$

$$\text{Solving, } \frac{H(z)}{z} = \frac{(z + 1)^2}{z(z - 0.5)(z + 0.75)}$$

$$\frac{(z + 1)^2}{z(z - 0.5)(z + 0.75)} = \frac{A}{z} + \frac{B}{z - 0.5} + \frac{C}{(z + 0.75)}$$

$$(z + 1)^2 = A(z - 0.5)(z + 0.75) + B(z)(z + 0.75) + C(z)(z - 0.5)$$

$$= \frac{(z + 1)^2}{(z + 0.75)(z - 0.5)} \Big|_{z=0}$$

$$\therefore A = \frac{1}{-0.375} = -2.66 = -\frac{8}{3}$$

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$$B = (z - 0.5) \times \frac{H(z)}{z} \Big|_{z=0.5}$$

$$= (\cancel{z - 0.5}) \cdot \frac{(z+1)^2}{z(\cancel{z - 0.5})(z+0.75)}$$

$$= \frac{2.25}{0.625}$$

$$= 3.6 = \frac{18}{5}$$

$$C = (z + 0.75) \times \frac{H(z)}{z} \Big|_{z=-0.75}$$

$$= (\cancel{z + 0.75}) \times \frac{(1+z)^2}{z(\cancel{z + 0.75})(z-0.5)}$$

$$= \frac{0.0625}{0.9375}$$

$$= 0.066 = \frac{1}{15}$$

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$$\text{Now, } \frac{H(z)}{z} = \frac{-8}{3z} + \frac{18}{5} \left(\frac{1}{z-0.5} \right) + \frac{1}{15} \left(\frac{1}{z+0.75} \right)$$

$$H(z) = -\frac{8}{3} + \frac{18}{5} \left(\frac{z}{z-0.5} \right) + \frac{1}{15} \left(\frac{z}{z+0.75} \right)$$

$$= -\frac{8}{3} + \frac{18}{5} \left(\frac{1}{1-0.5z^{-1}} \right) + \frac{1}{15} \left(\frac{1}{1+0.75z^{-1}} \right)$$

Taking z^{-1} transform,

$$h(n) = -\frac{8}{3} \delta(n) + \frac{18}{5} (0.5)^n u(n) + \frac{1}{15} (0.75)^n u(n)$$

Note: #

$$\begin{aligned} z^{-1} \delta(n) &= \sum_{n=-\infty}^{\infty} \delta(n) z^{-1} \\ &= \sum_{n=-\infty}^{\infty} \delta(n) z^0 \quad \left(\boxed{+1} \right) \end{aligned}$$

$\delta(n)$ r.

Q) $x(n) = 4^n$

$$3y(n) = 4y(n-1) - y(n-2) + \frac{x(n) + y(n-1)}{4}$$

Taking z-transform,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{4}z^{-1}}{3 - 4z^{-1} + z^{-2}}$$

$$H(z) = \frac{z(z + \frac{1}{4})}{(3z-1)(z-1)} \quad \dots \text{--- (i)}$$

Similarly, $y(z) =$

Ans,

$$y(z) = x(z) \cdot H(z) = \left(\frac{z}{z-1}\right) \cdot \frac{z(z + \frac{1}{4})}{(3z-1)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{z(z + \frac{1}{4})}{(3z-1)(z-1)^2} = \frac{A}{(3z-1)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}$$

Equating and find the value of A, B, and C,

$$A = (3z-1) \cdot \frac{Y(z)}{z}$$

$$y(z) =$$

$$= \frac{x \cdot 1}{(1 - \frac{1}{2}z^{-1})} + \left(\frac{1}{1-z^{-1}}\right) + \frac{z^{-1}}{(1-z^{-1})^2}$$

We know,

$$\begin{aligned}
 y(n) &\xrightarrow{z} \frac{1}{1-z^{-1}} \\
 n \cdot y(n) &\xrightarrow{z} -z \cdot \frac{d}{dz} \left(\frac{1}{1-z^{-1}} \right) \\
 &= -z \frac{d}{dz} (1-z^{-1})^{-1} \\
 &= -z (-1)(1-z^{-1})^{-2} (-1) \times (-1) z^{-2} \\
 &= \frac{z^{-1}}{(1-z^{-1})^2}
 \end{aligned}$$

Now,
Taking z^{-1} transform of eqn (i):

$$y(n) = \frac{1}{3} \left(\frac{1}{3}\right)^n u(n) + u(n) + n \cdot y(n)$$

#

IIR filter Design:

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Infinite Impulse Response:

Response $y(n)$: $h(n)$

- The system having infinite impulse response $h(n)$ is called IIR system.
- All the IIR system are ^{zero} ~~Poll~~ System
- IIR system are recursive in nature.
- IIR system can not be describe by linear convolution.

IIR filter realization method:

- ① Direct form → Direct form I
" " II
- (2) Cascade Realization.
- (3) Parallel " "
- Lattice structure realization.

General Expression for IIR filter system.

Consider a LCD equation as.

$$y(n) = - \sum_{k=L}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k) \quad \text{--- (i)}$$

Taking z-transform we get,

$$y(z) = - \sum_{k=L}^N a_k z^{-k} \cdot y(z) + \sum_{k=0}^M b_k z^{-k} x(z)$$

or
$$y(z) \left[1 + \sum_{k=L}^N a_k z^{-k} \right] = \sum_{k=0}^M b_k z^{-k} x(z).$$

$$\therefore H(z) = \frac{y(z)}{x(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=L}^N a_k z^{-k}} \quad \text{--- (ii)}$$

Eqⁿ (ii) is the general expression for the system function, of IIR filters.

Where $N \geq M$ is the order of the IIR filter

a_k = poles

b_k = zeros

① Direct form I - Realization.

We know,

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=L}^N a_k z^{-k}} \quad \text{--- (iii)}$$

or,
$$H(z) = H_1(z) \cdot H_2(z) = \frac{h_1(z)}{x(z)} \cdot \frac{y(z)}{x(z)}$$

$$= \sum_{k=0}^M b_k z^{-k}$$

$$1 + \sum_{k=L}^N a_k z^{-k}$$

Let,

$$H_1(z) = \frac{W(z)}{X(z)} = \sum_{k=0}^M b_k z^{-k} \text{ and}$$

$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 + \sum_{k=L}^N a_k z^{-k}}$$

Now, Taking $H_1(z)$, we have,

$$\frac{W(z)}{X(z)} = \sum_{k=0}^M b_k z^{-k}$$

$$W(z) = \sum_{k=0}^M b_k z^{-k} \cdot X(z) \quad \text{--- (i)}$$

Now, taking z^{-1} transform of eqn (i),

$$w(n) = \sum_{k=0}^M b_k x(n-k)$$

$$= b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \dots$$

$$b_M x(n-M) \quad \text{--- (iii)}$$

Note: $x(n) \xrightarrow{z^{-1}} x(n-1) \xrightarrow{z^{-1}} x(n-1-1) \dots x(n-1-1)$
memory element.

Now, eqn (iii) can be graphically represents,

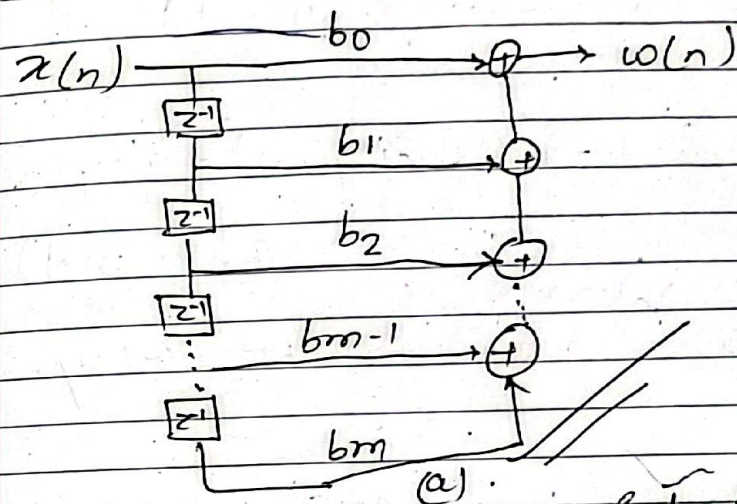


Fig: Direct form I realization of $H_1(z)$

Again, Taking $H_2(z)$,

$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 + \sum_{k=L}^N a_k z^{-k}}$$

$$Y(z) \left[1 + \sum_{k=L}^N a_k z^{-k} \right] = W(z)$$

Taking z^{-1} transform;

$$y(n) + \sum_{k=1}^N a_k y(n-k) = w(n)$$

$$y(n) = w(n) - \sum_{k=1}^N a_k y(n-k)$$

$$= w(n) - [a_1 y(n-1) + a_2 y(n-2) + \dots + a_N y(n-N)]$$

Eq (iv) can be graphically represented by, (iv)

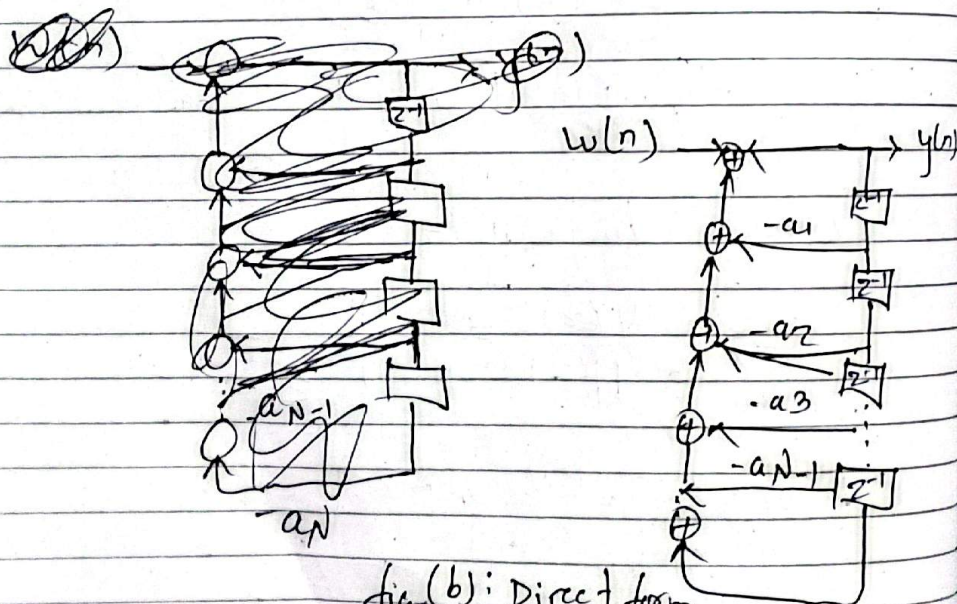


fig (b): Direct form-I Realization of $H(z)$

finally combined figure (a) (b).

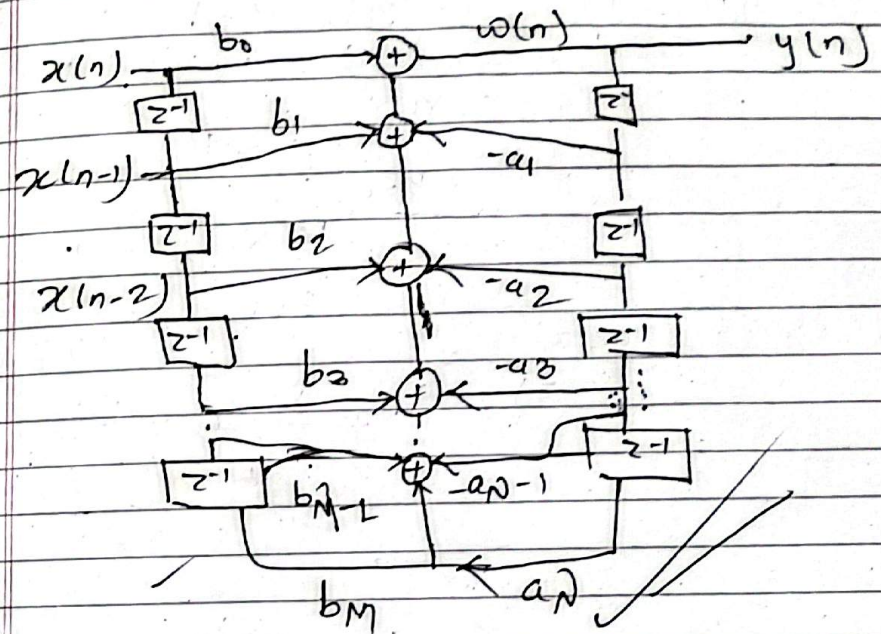


Fig: Direct form-I structure

→ Computational Complexity of Direct form-I

- ① It requires $N+M$ no. of memory element i.e. z^{-1} item
- ② It requires $N+M+1$ no. of addition.

Direct form - II structure:

$$H(z) = H_1(z) \cdot H_2(z) = \frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)}$$

$$= \frac{\sum_{k=0}^m b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

$$\text{let } H_1(z) = \frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^N a_k z^{-k}}$$

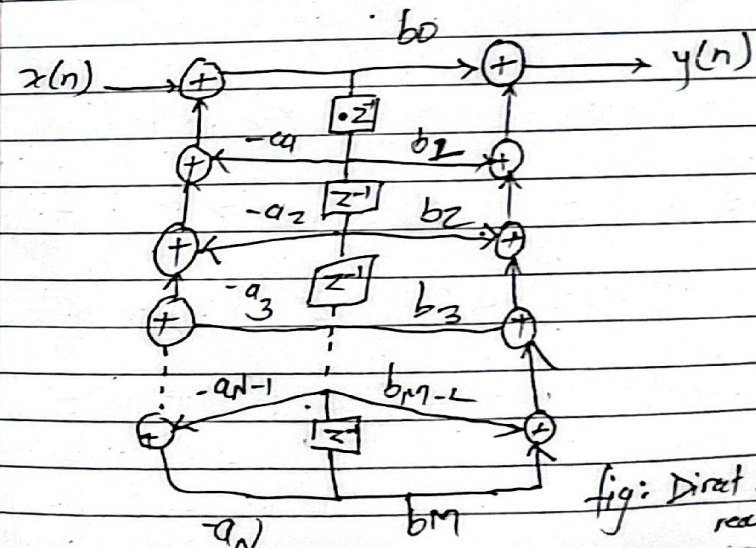


fig: Direct form - II realization.

Computational complexity of Direct form - II.

- ① It has $(N+M)$ memory elements
- ② " " $(N+M+1)$ addition.

Q. Why direct form II is preferred over direct form - I? Obtain direct form II realization for the system
 $y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6z(n-1) + 0.6z(n-2)$

⇒ Direct form - II is preferred over Direct form - I as direct form - I requires either N or M memory element. Where as direct form II requires $N+M$ memory element.

Taking z transform on both side,

$$y(z) = -0.1z^{-1}y(z) + 0.2z^{-2}y(z) + 3x(z) + 3.6z^{-1}x(z) + 0.6z^{-2}x(z)$$

$$y(z) + 0.1z^{-1}y(z) - 0.2z^{-2}y(z) = 3x(z) + 3.6z^{-1}x(z) + 0.6z^{-2}x(z)$$

$$y(z) \left[1 + 0.1z^{-1} - 0.2z^{-2} \right] = x(z) \left[3 + 3.6z^{-1} + 0.6z^{-2} \right]$$

$$\frac{y(z)}{x(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} \quad \text{--- (i)}$$

Comparing with general expression of IIR filter,

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \quad \text{--- (ii)}$$

Now, comparing eqn (i) & (ii) we get,

$$\begin{aligned} b_0 &= 3 & a_1 &= 0.1 \\ b_1 &= 3.6 & a_2 &= -0.2 \\ b_2 &= 0.6 \end{aligned}$$

Direct form II structure,

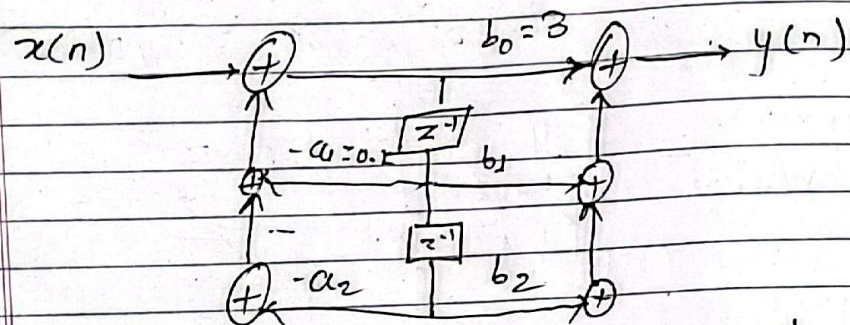


fig: Direct form II realization

Direct form I structure

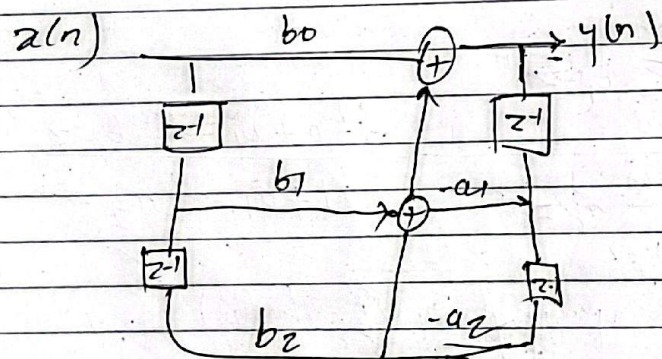


fig:

8.] $H(z) = \frac{(1+z^{-1})^2}{(1-0.5z^{-1})(1+0.75z^{-1})}$

obtain Direct form -I and II structure.

$\Rightarrow H(z) = \frac{1 + 2 \cdot z^{-1} + z^{-2}}{1 - 0.5z^{-1} + 0.75z^{-1} - 0.375z^{-2}}$

$H(z) = \frac{1 + 2 \cdot z^{-1} + z^{-2}}{1 + 0.25z^{-1} - 0.375z^{-2}}$

Comparing with general expression of IIR filter

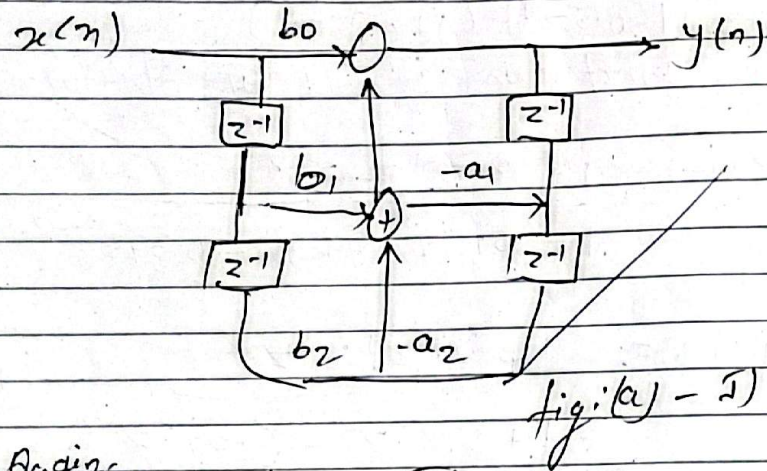
$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$ (1)

Now, Comparing eqn (1) to get.

$b_0 = 1, b_1 = 2, b_2 = 1$

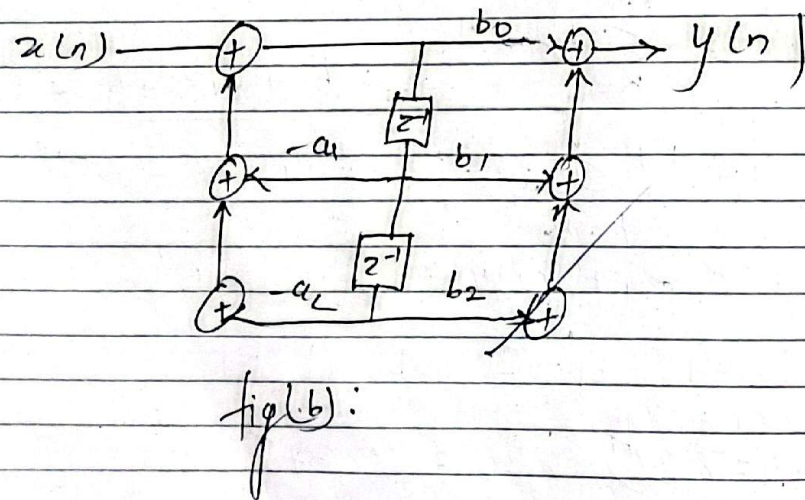
$a_1 = -0.25, a_2 = 0.375$

Now, Structure - I,



Again,

Structure - (II)



$$H(z) = \frac{z(z+1)}{(z-1)\left(z+\frac{1}{4}\right)}$$

Obtain Direct form - I and Direct form - II Structure.

$$\begin{aligned} H(z) &= \frac{z(z+1)}{(z-1)\left(z+\frac{1}{4}\right)} \\ &= \frac{z^2(1+z^{-1})}{(z^2 - \frac{3}{4}z + \frac{1}{4})} \\ &= \frac{z^2(1+z^{-1})}{z^2\left(1 - \frac{3}{4}z^{-1} + \frac{1}{4}z^{-2}\right)} \end{aligned}$$

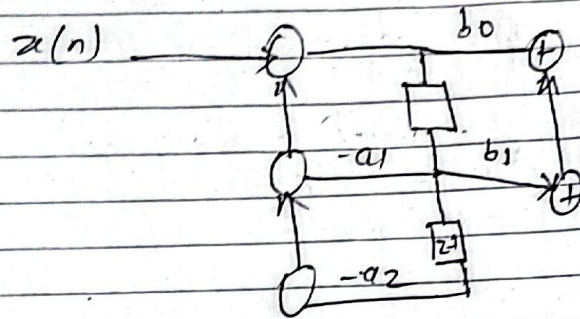
Comparing general expression.

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} \quad \text{--- (i)}$$

Comparing,

$$b_0 = 1, b_1 = 1, a_1 = -\frac{3}{4}, a_2 = \frac{1}{4}$$

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Where, $\eta_1(z), \eta_2(z), \dots, \eta_p(z)$ all are ~~form~~ represented by ~~the~~ direct form-II realization.

$$\omega(n)$$

Q3] Obtain cascade realization, for
 $y(n) = 3y(n-1) - 2y(n-2) + x(n)$

⇒ Taking z-transform,

$$Y(z) = 3z^{-1}Y(z) - 2z^{-2}Y(z) + X(z)$$

$$Y(z) [1 - 3z^{-1} + 2z^{-2}] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 3z^{-1} + 2z^{-2}}$$

$$H(z) = \frac{1}{2z^{-2} - 3z^{-1} + 1}$$

$$= \frac{1}{2z^{-2} - 2z^{-1} - z^{-1} + 1}$$

$$= \frac{1}{2z^{-1}(z^{-1}-1) - 1(z^{-1}-1)}$$

$$= \frac{1}{(z^{-1}-1)} \times \frac{1}{(2z^{-1}-1)}$$

$$= H_1(z) \cdot H_2(z)$$

$$H_1(z) = \frac{1}{(1-z^{-1})}, \quad H_2(z) = \frac{1}{(1-2z^{-1})}$$

$$\text{So, } H(z) = H_1(z) \cdot H_2(z)$$

$$H(z) = \frac{1}{(1-z^{-1})} \cdot \frac{1}{(1-2z^{-1})} \quad \text{--- (I)}$$

Comparing general eqn (I),

$$H(z) = \frac{\sum_{k=0}^m b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \quad \text{--- (II)}$$

Equating, $b_0 = 1, a_1 = 1$ for $H_1(z)$

for $H_2(z)$, $b_0 = 1, a_1 = 2$ (2)

$x(n]$

Where $h_1(z), h_2(z) \dots h_p(z)$ all are respectively by direct form - II realization represented

and,

$h_1(z), h_2(z) \dots h_p(z)$ are obtain by parallel fraction.

Q7. Obtain parallel realization of

$$H(z) = \frac{z(z+1)}{(z-1)(z-\frac{1}{4})}$$

$$\Rightarrow \frac{H(z)}{z} = \frac{z+1}{(z-1)(z-\frac{1}{4})} = \frac{A}{(z-1)} + \frac{B}{(z-\frac{1}{4})}$$

By partial fraction,

Where

$$A = (z-1) \cdot \frac{H(z)}{z} \Big|_{z=1}$$

$$= (z-1) \cdot \frac{(z+1)}{(z-1)(z-\frac{1}{4})}$$

$$= \frac{8}{3}$$

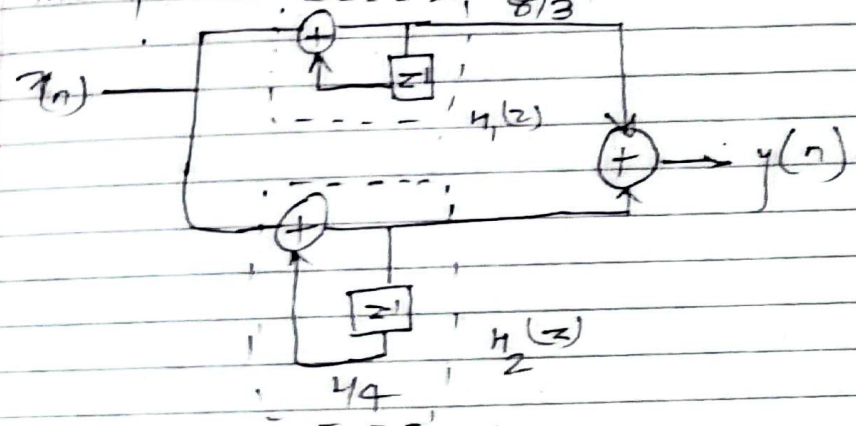
$$B = (z-\frac{1}{4}) \cdot \frac{H(z)}{z} \Big|_{z=\frac{1}{4}} = \frac{-5}{3}$$

$$\frac{H(z)}{z} = \frac{8/3}{(z-1)} - \frac{5/3}{(z-\frac{1}{4})}$$

$$H(z) = \frac{8/3}{1-z^{-1}} - \frac{5/3}{1-\frac{1}{4}z^{-1}}$$

$$\therefore H(z) = H_1(z) - H_2(z)$$

Now the parallel realization of $H(z)$ is.



Q7] $x(n) = 2 \left(\frac{5}{6} \right)^n u(n-1) + 3 \left(\frac{1}{2} \right)^n u(n)$.
Find the parallel realization $H(z)$.

⇒ Taking z -transform.

$$H(z) = 2 \cdot \frac{\frac{5}{6} z^{-1}}{1 - \frac{5}{6} z^{-1}} + \frac{3/2}{1 - \frac{1}{2} z^{-1}}$$

we have,

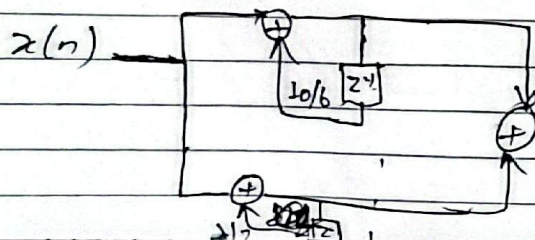
$$u(n) \xrightarrow{z} \frac{1}{1 - z^{-1}}$$

$$u(n-1) \xrightarrow{z} \frac{z^{-1}}{1 - z^{-1}}$$

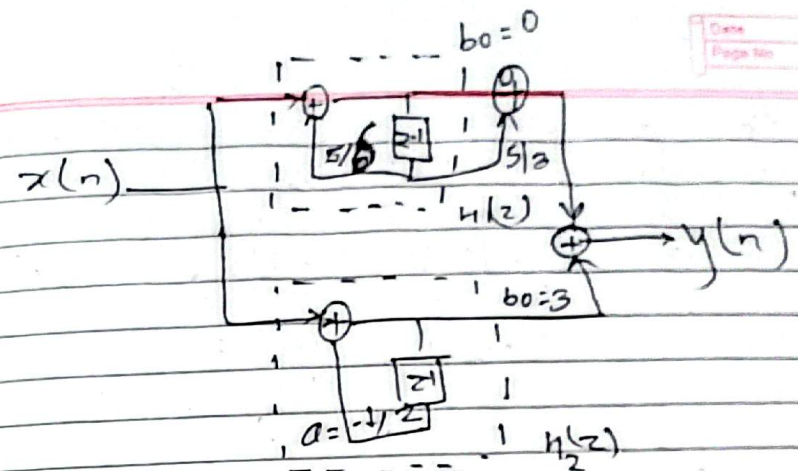
So zero.

$$H(z) = \frac{10/6 z^{-1}}{1 - \frac{5}{6} z^{-1}} + \frac{3/2}{1 - \frac{1}{2} z^{-1}}$$

$$H(z) = H_1(z) + H_2(z).$$



or,



Q7] Parallel realization of $H(z)$.

$$H(z) = \frac{z+1}{(z-1)(z-\frac{1}{4})}$$

$$\frac{H(z)}{z} = \frac{(1 + \frac{1}{z})}{z(z-1)(z-\frac{1}{4})} = \frac{A}{z(z-1)} + \frac{B}{(z-\frac{1}{4})}$$

By partial fraction,

$$A = \left. \frac{(z-\frac{1}{4}) \cdot H(z)}{z} \right|_{z=0}$$

$$= \frac{(z-1)(1+\frac{1}{z})}{(z-\frac{1}{4})} \Big|_{z=0} = \frac{8}{3}$$

$$B = \frac{(z-1) \cdot \frac{H(z)}{z}}{z} \Big|_{z=1/4}$$

$$= \frac{(z-1/4) \cdot (1 + \frac{1}{z})}{(z-1)(z-1/4)}$$

$$= \cancel{z} \cdot \frac{(z+1)}{\cancel{z}(z-1)(z-1/4)}$$

$$= \frac{1}{1/4}$$

$$= 4$$

$$B = (z-1) \frac{H(z)}{z} \Big|_{z=1}$$

$$= (z-1) \cdot \frac{(z+1)}{z(z-1)(z-1/4)}$$

$$= \frac{2}{3/4}$$

$$= 8/3$$

$$C = \frac{(z-1/4) \cdot \frac{H(z)}{z}}{z} \Big|_{z=1/4}$$

$$= \frac{(z-1/4) \cdot (z+1)}{z(z-1)(z-1/4)}$$

$$= \frac{5/4}{\frac{1}{4} \times (-3/4)}$$

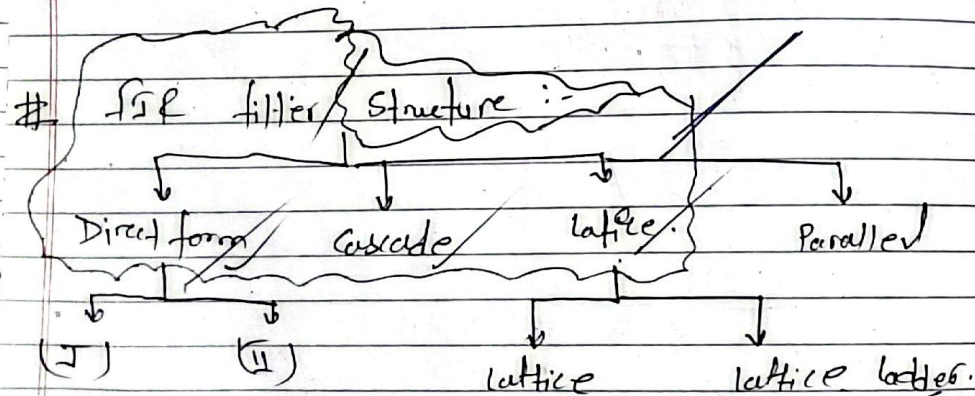
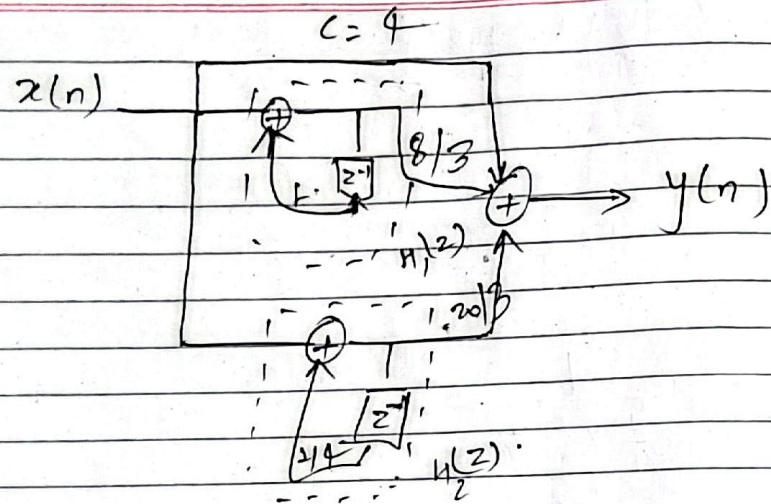
$$= \frac{5/4}{-3/16}$$

$$= -20/3$$

Now, $\frac{H(z)}{z} = \frac{4}{z} + \frac{8/3}{(z-1)} - \frac{20/3}{(z-1/4)}$

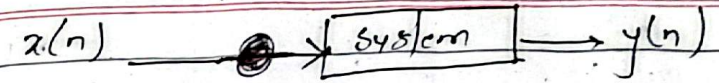
$$H(z) = 4 + \frac{8/3}{(1-z^{-1})} - \frac{20/3}{(1-\frac{1}{4}z^{-1})}$$

Now,



FIR filter :-

- finite impulse response (i.e. $h(n)$ is finite)
- All zeros system
- No feedback path
- No recursive in nature



→ linear convolution can be applied.

General expression of FIR filter :-
Consider LCCD equation as,

$$y(n) = \sum_{k=0}^M b_k x(n-k) \quad \text{--- (i)}$$

or, $y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \dots + b_M x(n-M)$ --- (ii)

Eqn (ii) can be expressed/represented graphically as

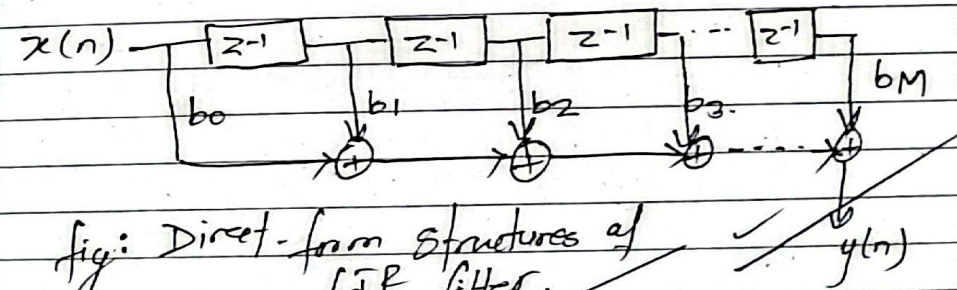


fig: Direct-form structures of FIR filter.

Where,

b_k is the location of the zeros.

Now, taking z-transform of eqn (i), we get.

$$Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^M b_k z^{-k} \quad \text{--- (iii)}$$

Eqn (iii) is the transfer function of FIR filter.

From Eqn (iii) it is clear that,

b_k is the impulse response $h(n)$ of the FIR filter.
 $\therefore$ FIR filter can be represented as,

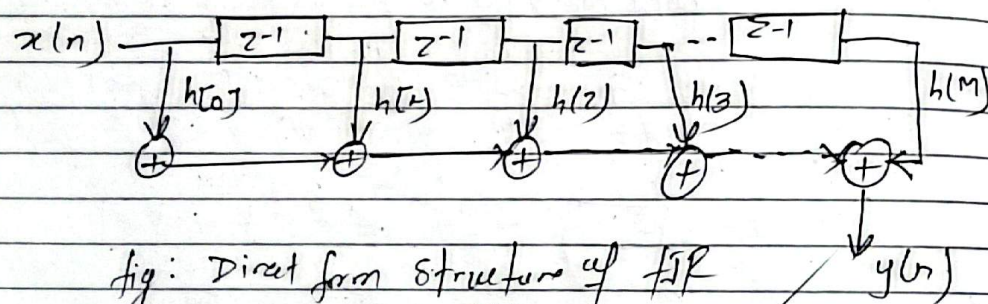


fig: Direct form structure of FIR filter.

Q7] Obtain the direct form realization of
 $y(n) = \frac{1}{2}x(n) + \frac{1}{4}(x(n-1) + \frac{1}{3}x(n-2) + x(n-3))$

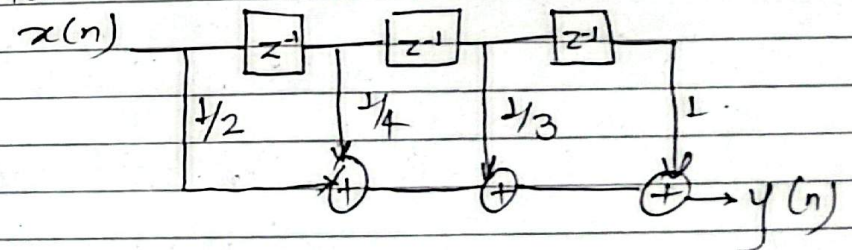
$\Rightarrow$ Taking z-transform,

$$Y(z) = \frac{1}{2}X(z) + \frac{1}{4}z^{-1}X(z) + \frac{1}{3}z^{-2}X(z) + z^{-3}X(z)$$

$$Y(z) = X(z) \cdot \left(\frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{3}z^{-2} + z^{-3} \right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{3}z^{-2} + z^{-3}$$

Here,



Q8] Obtain the direct form structure of $h(n) = \{1, \frac{1}{2}, \frac{1}{4}, 2\}$

$\Rightarrow$ it's corresponding $H(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + 2z^{-3}$

$\downarrow$... are same as above

Cascade Realization of FIR filter.

$$H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z) \dots H_N(z)$$

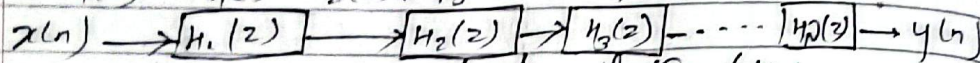


fig: cascade realization of FIR filter.

Where, $H_1(z), H_2(z) \dots H_N(z)$ all are represented by direct form structure.

Q. Obtain the cascade structure of $y(n) = x(n) + 3x(n-1) + 2x(n-2)$

⇒ Taking z-transform,

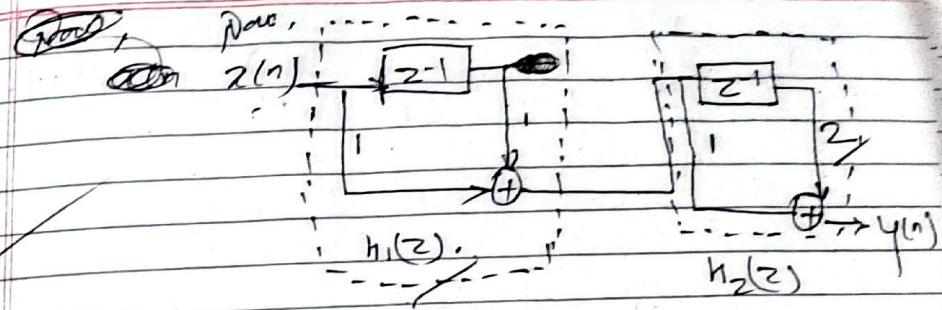
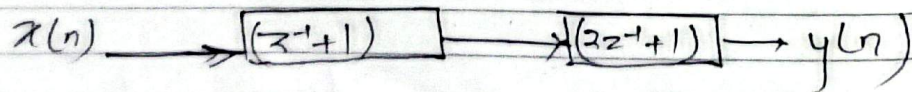
$$Y(z) = X(z) + 3z^{-1}X(z) + 2z^{-2}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + 3z^{-1} + 2z^{-2}$$

$$= 1 + 2z^{-1} + z^{-1} + 2z^{-2}$$

$$= (1 + 2z^{-1}) (1 + z^{-1})$$

$$= (z^{-1} + 1)(2z^{-1} + 1) = H_1(z) \cdot H_2(z)$$



Lattice Structure of FIR filter:

We know, for FIR filter,

$$H(z) = \sum_{n=0}^N h(n) z^{-n} = \sum_{n=0}^N b_n z^{-n}$$

$$= \sum_{n=0}^N a_n(n) z^{-n}$$

Where $h(n) \rightarrow$ Impulse response
 $b_n \rightarrow a_n(n) \rightarrow$ location of zeros.

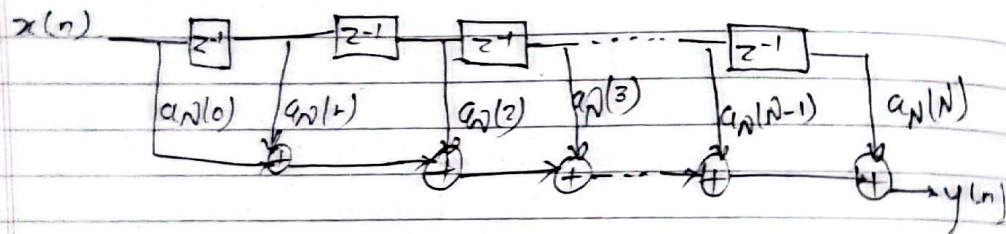
$$\text{or, } H(z) = a_N(0) + a_N(1)z^{-1} + a_N(2)z^{-2} + \dots + a_N(N)z^{-N}$$

Where $a_N(0) = 1$ (always)

$a_N(N) = k_N$; Where k_N is called lattice coefficient

Also, $a_N(0), a_N(1), a_N(2), \dots, a_N(N)$ are called direct form coefficient of FIR filter.

The direct form structure of a_N^n is,



The lattice structure will be represented as

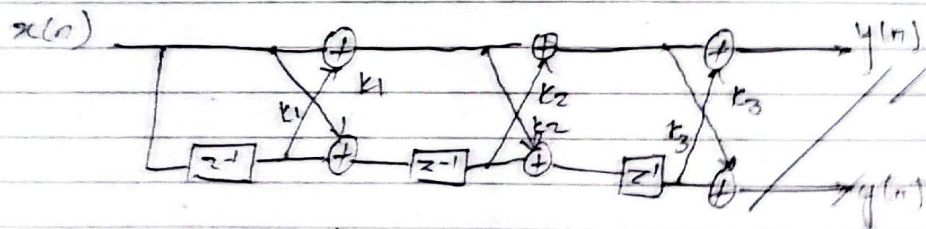


fig: 3 stage lattice structure of FIR filter.

Where, k_1, k_2, k_3 are the lattice coefficients and

$$k_1 = a_1(1)$$

$$k_2 = a_2(2)$$

$$k_3 = a_3(3)$$

for stability $|k_n| < 1$.

It for the conversion between direct form and lattice coefficient

(1) Direct to lattice coefficient conversion

$$a_{m-1}(k) = a_m(k) - a_m(m) a_m(m, k)$$

$1 - a_m(m)$ lattice coefficient

(2) Lattice coefficients to direct form coefficient

$$a_m(k) = a_{m-1}(k) + a_m(m) a_m(m, k)$$

Ex

$$h(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}$$

lattice structure and convert the lattice structure to the direct form structure

Ex

$$h(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}$$

here

now, for FIR filter,

$$h(z) = \sum_{n=0}^N a(n) z^{-n}$$

$$= \sum_{n=0}^2 a(n) \cdot z^{-n}$$

$$= a_2(0) + a_2(1) z^{-1} + a_2(2) z^{-2} \quad (ii)$$

from (i) and (ii),

$$a_2(0) = 1,$$

$$a_2(1) = 7/9$$

$$a_2(2) = 3/5 = k_2$$

for, Direct to lattice coefficient conversion,

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m) \cdot a_m(m-k)}{1 - a_m(m)} \quad (iii)$$

$$N=2, m=2, 2$$

$$\text{for } m=2, k=1$$

$$\therefore a_1(1) = \frac{a_2(1) - a_2(2) \cdot a_2(2-1)}{1 - a_2(2)}$$

$$\begin{aligned} a_1(1) &= \frac{7/9 - 3/5 \times 7/9}{1 - (3/5)^2} \\ &= \frac{7/9 - 21/45}{1 - 9/25} \\ &= 63/80 \end{aligned}$$

$\therefore$ The lattice coefficient are,

$$k_1 = 1$$

$$k_2 = 3/5$$

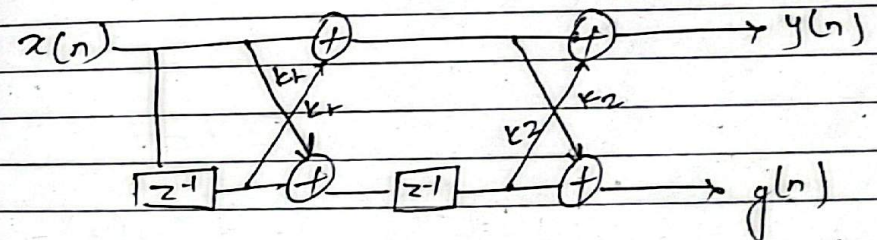


fig: Two stage lattice structure.

Since, $|k_m| < 1$, Hence the lattice structure is stable.

Discribe by LCD structure

Date
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8] $y(n) = x(n) + \frac{13}{24} x(n-1) + \frac{5}{8} x(n-2) + \frac{1}{3} x(n-3)$

⇒ Taking z-transform on both sides,

$$Y(z) = X(z) + \frac{13}{24} z^{-1} X(z) + \frac{5}{8} z^{-2} X(z) + \frac{1}{3} z^{-3} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}$$

Here, $N=3$
FIR filter,

$$H(z) = \sum_{n=0}^N a_n z^{-n}$$

$$= \sum_{n=0}^3 a_n z^{-n}$$

$$= a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} \quad \text{--- (ii)}$$

from eqn (i) and (ii),

$$a_3(0) = 1$$

$$a_3(1) = 13/24$$

$$a_3(2) = 5/8$$

$$a_3(3) = 1/3 = k_3$$

$$N=3, \quad m=3, 2, 1$$

$$\text{for, } m=3, \quad k=2, 1$$

for, $m=3, \quad k=2$, then,

$$a_2(2) = \frac{a_3(2) - a_3(3) \cdot a_3(1)}{1 - a_3^2(3)}$$

$$= \frac{5/8 - 1/3 \times 13/24}{1 - (1/3)^2}$$

$$= \frac{5/8 - 13/72}{1 - 1/9} = \frac{1}{2} \quad (\text{Ans})$$

$$1 - \frac{1}{9} = \frac{8}{9}$$

$$= \frac{360 - 104}{576}$$

$$= \frac{256}{576}$$

$$= \frac{16 \times 16}{36 \times 16}$$

$$= \frac{16}{36} = \frac{4}{9}$$

$$= \frac{16 \times 16}{36 \times 16} = \frac{256}{576}$$

$$= \frac{16}{36} = \frac{4}{9}$$

for, $m=3, \quad k=1$

$$a_2(1) = \frac{a_3(1) - a_3(3) \cdot a_3(2)}{1 - a_3^2(3)}$$

$$a_2(1) = \frac{13/24 - 1/3 \times 5/8}{1 - (1/3)^2}$$

$$= \frac{13/24 - 5/24}{9-1}$$

$$= \frac{13-5}{24} = \frac{8}{24} = \frac{1}{3}$$

Again,

for, $m=2, k=1, /$

$$k_1 = a_1(1) = \frac{a_2(1) - a_2(2) \cdot a_2(1)}{1 - a_2(2)}$$

$$= \frac{3/8 - 1/2 \times 3/8}{1 - (1/2)^2}$$

$$= \frac{3/8 - 3/16}{3/4} = \frac{6-3}{16} = \frac{3 \times 4}{8 \times 16} = \frac{3}{4} = \frac{1}{4}$$

$\therefore$ The lattice coefficient.

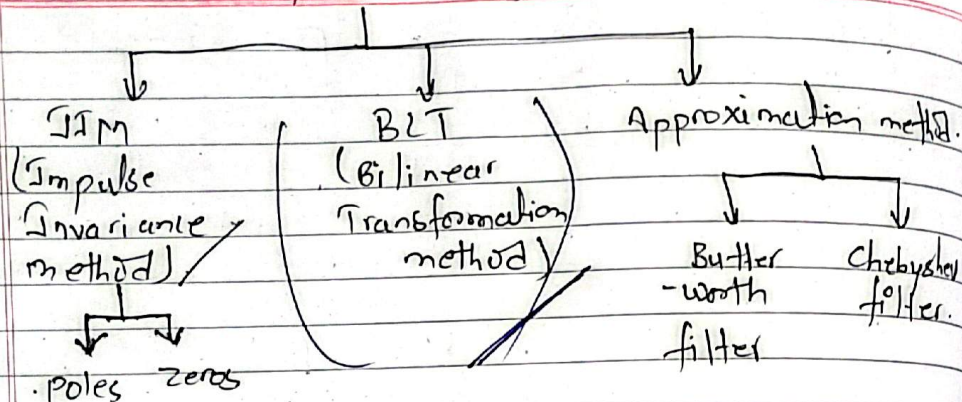
$$k_1 = 1/4$$

$$k_2 = 1/2$$

$$k_3 = 1/3$$

Unit:

ISR filter Design:-



$H(s)$ = Analog transfer function.
 s = Laplace operator; $s = \sigma + j\omega$
 ω = Analog frequency.
 w = Digital frequency.

$H(z)$ = Digital filter Transfer function
 $z = re^{j\omega}$

ISR filter Design by ITM:
 Consider an analog system having transfer function $H(s) = \sum_{i=1}^N \frac{A_i}{s - p_i}$

Where,
 $\{A_i\}$ = constants obtained using partial fraction
 $\{p_i\}$ = location of poles.

Now, taking inverse Laplace transform of eqⁿ (i),

$$h_a(t) = \sum_{i=1}^N A_i e^{p_i t} \quad \text{--- (ii)}$$

Now, using sampling theorem, we have,

$$h[n] = h_a(t) \Big|_{t=nT} \quad \text{Where } T \text{ is sampling interval.}$$

Hence, eqⁿ (ii) can be written as,

$$h[n] = h_a(t) \Big|_{t=nT}$$

$$h[n] = \sum_{i=1}^N A_i e^{p_i nT} \quad \text{--- (iii)}$$

Now, taking z-transform of eqⁿ (iii),

$$\begin{aligned}
 H(z) &= \sum_{n=0}^{\infty} h[n] \cdot z^{-n} \\
 &= \sum_{n=0}^{\infty} \sum_{i=1}^N A_i (e^{p_i T})^n \cdot z^{-n}
 \end{aligned}$$

Interchanging the order of summation

$$H(z) = \sum_{i=1}^N A_i^0 \sum_{n=0}^{\infty} (e^{j\pi i T} z^{-1})^n$$

$$= \sum_{i=1}^N A_i^0 \left[1 + (e^{j\pi i T} z^{-1}) + (e^{j\pi i T} z^{-1})^2 + \dots \right]$$

$$= \sum_{i=1}^N A_i^0 \times \frac{1}{1 - e^{j\pi i T} z^{-1}}$$

$$\therefore H(z) = \sum_{i=1}^N \left(\frac{A_i^0}{1 - e^{j\pi i T} z^{-1}} \right) \quad \text{--- (iv)}$$

Comparing Eqⁿ (iv) and (i), we get,

$$\therefore \sum_{i=1}^N \frac{A_i}{s - p_i} \xleftrightarrow{\text{ZTM}} \sum_{i=1}^N \frac{A_i}{1 - e^{j\pi i T} z^{-1}}$$

$$\therefore \frac{A_i}{s - p_i} \xleftrightarrow{\text{ZTM}} \frac{A_i}{1 - e^{j\pi i T} z^{-1}}$$

Similarly, $\frac{1}{s + p_i} \xleftrightarrow{\text{ZTM}} \frac{1}{1 - e^{-j\pi i T} z^{-1}}$

Additional formula for ZTM:

$$\frac{s+a}{(s+a)^2 + b^2} \xleftrightarrow{\text{ZTM}} \frac{1 - e^{aT} [\cos bT] z^{-1}}{1 - 2e^{aT} [\cos bT] z^{-1} + e^{2aT} z^{-2}}$$

$$\frac{b}{(s+a)^2 + b^2} \xleftrightarrow{\text{ZTM}} \frac{e^{-aT} [\sin bT] z^{-1}}{1 - 2e^{aT} [\cos bT] z^{-1} + e^{2aT} z^{-2}}$$

Frequency Mapping :-

We know, $z = e^{sT}$ (i) where T is sampling period.

Also, in Polar form;

$$z = re^{j\omega T} \quad \text{--- (ii)}$$

Now, from (i) and (ii),

$$e^{sT} = e^{j\omega T}$$

$$\text{or, } (e + j\Omega T) = j\omega$$

$$j\Omega T = j\omega$$

$$\therefore \Omega = \omega / T$$

Where, $\Omega \rightarrow$ Analog frequency, $\omega \rightarrow$ Digital frequency.

→ Frequency mapping: The process in which the analog frequency is converted into Digital frequency is called frequency mapping.

Q7) Obtain the digital transfer function for an analog system.

$$H(s) = \frac{1}{(s+2.5)(s^2 + 0.5s + 0.1)} \quad \text{by using IFT}$$

⇒ We know for IFT:-

$$\frac{1}{s-p} \xrightarrow{\text{IFT}} \frac{1}{1-e^{pT}z^{-1}}$$

$$\frac{1}{s+pi} \xrightarrow{\text{IFT}} \frac{1}{1-e^{pT}z^{-1}}$$

$$\frac{s+a}{(s+a)^2 + b^2} \xrightarrow{\text{IFT}} \frac{1 - e^{-aT} \cos(bT)z^{-1}}{1 - 2e^{-aT} \cos(bT)z^{-1} + e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2 + b^2} \xrightarrow{\text{IFT}} \frac{e^{-aT} \sin(bT)z^{-1}}{1 - 2e^{-aT} \cos(bT)z^{-1} + e^{-2aT}z^{-2}}$$

$$\frac{1}{(s+2.5)(s^2 + 0.5s + 0.1)} = \frac{A}{s+2.5} + \frac{B}{s+0.1} + \frac{C}{s+0.5}$$

By partial fraction

$$1 = A(s+2.5)(s^2 + 0.5s + 0.1) + B(s+2.5)(s+0.5) + C(s+2.5)(s+0.1)$$

$$1 = (A+B)s^2 + (2.5A + 0.5B + C)s + 0.25A + 0.125B + 0.025C$$

Equating $A+B=0$
 $2.5A + 0.5B + C = 1$
 $0.25A + 0.125B + 0.025C = 0$

$$2A + 0.5C = 1 \quad C = 2 - 4A$$

$$0.5A + 0.5B + 1 = 0$$

$$0.5A - 0.5A + 1 = 0$$

$$1 = 0$$

$$A = 0.5, \quad B = -0.5$$

Now,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{s^2+0.5s+2}$$

$$= \frac{0.5}{s+0.5} - \frac{0.5s}{\left(s^2 + 2\left(\frac{1}{4}\right)s + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + 2\right)}$$

$$= \frac{0.5}{s+0.5} - \frac{0.5s}{\left(s + \frac{1}{4}\right)^2 + 3/16}$$

$$= \frac{0.5}{s+0.5} - \frac{0.5s}{(s+0.25)^2 + (1.392)^2}$$

$$= \frac{0.5}{s+0.5} - \left\{ \frac{0.5(s+0.25-0.25)}{(s+0.25)^2 + (1.392)^2} \right\}$$

$$= \frac{0.5}{s+0.5} - \frac{0.5(s+0.25)}{(s+0.25)^2 + (1.392)^2} + \frac{0.125}{(s+0.25)^2 + (1.392)^2}$$

$$= \frac{0.5}{s+0.5} - \left\{ \frac{0.5(s+0.25)}{(s+0.25)^2 + (1.392)^2} \right\} + \frac{0.125}{1.392} \left\{ \frac{1.392}{(s+0.25)^2 + (1.392)^2} \right\}$$

Ⓘ

from eqn (i).

$$p = 0.5 \quad \left| \begin{array}{l} a = 0.25 \\ b = 1.392 \end{array} \right.$$

Now applying SIM in eqn (i).

$$H(z) = 0.5 \frac{1}{1 - e^{0.5} z^{-1}} - 0.5 \left(\frac{1 - e^{0.25} \cos(1.392) z^{-1}}{1 - 2e^{0.25} \cos(1.392) z^{-1} + e^{2 \times 0.5} z^{-2}} \right)$$

$$+ 0.089 \left(\frac{e^{-0.25} \sin(1.392) z^{-1}}{1 - 2e^{-0.25} \cos(1.392) z^{-1} + e^{-2 \times 0.5} z^{-2}} \right)$$

Assume $T = 1 \text{ sec}$ [if sampling frequency was given then we use $T = \frac{1}{f_s} \rightarrow \text{sampling frequency}$]

Then,

$$H(z) = \frac{0.5}{1 - 0.606 z^{-1}} - 0.5 \left(\frac{1 - 0.707 z^{-1}}{1 - 1.414 z^{-1} + 0.368 z^{-2}} \right)$$

Note:

1. → for IIM, analog frequency $\Omega = \omega/T$
2. → convert the analog transfer function in $(s+p_i)(s-p_i)$ by factorizing then apply practical function

QJ. obtain digital transfer function for the analog system.

$$H_a(s) = \frac{1}{(s+4)^2 + 16}$$

resonance $\omega_r = \pi/2$ (resonant frequency / cut off freq. / 3dB freq. / half power point frequency.)

⇒ we know, for IIM, $\Omega = \omega/T$

As we know, $s = \sigma + j\omega$

from the denominator of eq (i) we get,

$$(s+4)^2 + 16 = 0$$

$$(s+4)^2 - (j4)^2 = 0$$

$$\therefore (s+4-j4)(s+4+j4) = 0$$

$$\therefore s = -4 + j4$$

$$\therefore s = \sigma + j\Omega = -4 + j4$$

$$\therefore \Omega = 4$$

from eq (ii), $\Omega = \omega/T$

$$\therefore T = \frac{\omega}{\Omega} = \frac{\omega_r}{\Omega} = \frac{\pi/2}{4} = \frac{\pi}{8}$$

Now, applying IIM, we get,

$$H_d(z) = \frac{1}{(s+4)^2 + 16} = \frac{1}{(s+4)^2 + 4^2}$$

Where,

$$\frac{b}{(s+a)^2 + b^2} \xrightarrow{\text{IIM}} \frac{e^{-aT} \sin(bT) z^{-1}}{1 - 2e^{-aT} (\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$H_a(s) = \frac{1}{4} \left(\frac{4}{(s+4)^2 + 4^2} \right)$$

$$= \frac{1}{4} \left[\frac{e^{-4T} \sin(4T) z^{-1}}{1 - 2e^{-4T} (\cos(4T) z^{-1} + e^{-2 \times 4T} z^{-2})} \right]$$

$$= \frac{1}{4} \left[\frac{e^{-4T} \sin(4T) z^{-1}}{1 - 2e^{-4T} (\cos(4T) z^{-1} + e^{-8T} z^{-2})} \right]$$

At $T = \pi/8$

$$= \frac{1}{4} \left[\frac{e^{-\pi/2} \sin(\pi/2) z^{-1}}{1 - 2e^{-\pi/2} (\cos(\pi/2) z^{-1} + e^{-\pi} z^{-2})} \right]$$

$$= \left[\frac{0.052 z^{-1}}{1 - 0.0482 z^{-2}} \right] \quad \#$$

Note: if sampling frequency is given then use.

$$T = \frac{1}{\text{Sampling frequency.}}$$

Q]. $h(z)$ find, $H_a(s) = \frac{s z_c}{s + s_c}$

having 3dB frequency $\pi/2$

Where s_c is the analog cut off frequency.

$\Rightarrow$ Given, $\omega_c = \pi/2$

$$s z_c = \frac{\omega_c}{T} = \frac{\pi/2}{T}$$

Let $T = 1 \text{ sec}$, $s_c = \pi/2$

i.e. $H_a(s) = \frac{\pi/2}{s + \pi/2}$

By IIM,

$$H(z) = \frac{\pi/2}{1 - e^{-\pi/2 T} z^{-1}}$$

$$= \frac{\pi/2}{1 - e^{-\pi/2} z^{-1}} \quad \#$$

#

SSR filter Design by Bilinear transformation method (BLT):

Consider an analog transfer function.

$$H_a(s) = \frac{b}{s+a}$$

$$\text{or, } H_a(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$\text{or, } sY(s) + a[Y(s)] = b[X(s)]$$

~~Now,~~ taking inverse Laplace transform, we get,

$$\frac{dy(t)}{dt} + a y(t) = b x(t)$$

Now,

integrating over one period, $(nT-T)$ to nT we get,

$$\int_{nT-T}^{nT} \frac{d y(t)}{dt} \cdot dt + a \int_{nT-T}^{nT} y(t) dt = b \int_{nT-T}^{nT} x(t) dt$$

Now, applying trapezoidal integration rule, in eqⁿ (i), we get

$$\text{i.e. } \int_{nT-T}^{nT} x(t) dt = \frac{T}{2} [x(nT) + x((nT-T))]]$$

eqⁿ (i) becomes,

$$y(nT) - y((nT-T)) + \frac{aT}{2} [y(nT) + y((nT-T))] = \frac{bT}{2} [x(nT) + x((nT-T))]$$

As,

$$y(nT) \rightarrow y[n]$$

$$\therefore y[(nT-T)] \rightarrow y[n-1]$$

$$\text{or, } y[n] - y[n-1] + \frac{aT}{2} [y[n] + y[n-1]] = \frac{bT}{2} [x[n] + x[n-1]]$$

Taking z-transform on both side, — (ii)

$$Y(z) - z^{-1} Y(z) + \frac{aT}{2} [Y(z) + z^{-1} Y(z)] = \frac{bT}{2} [X(z) + z^{-1} X(z)]$$

$$Y(z) [1 - z^{-1} + \frac{aT}{2} + \frac{aT}{2} z^{-1}] = X(z) [\frac{bT}{2} + \frac{bT}{2} z^{-1}]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(\frac{bT}{2} + \frac{bT}{2} z^{-1})}{(1 - z^{-1} + \frac{aT}{2} + \frac{aT}{2} z^{-1})}$$

$$= \frac{b \frac{1}{2} (1+z^{-1})}{(1-z^{-1}) + a \frac{1}{2} (1+z^{-1})}$$

$$= \frac{b \frac{1}{2} (1+z^{-1})}{(1-z^{-1}) + a \frac{1}{2} (1+z^{-1})} = \frac{b}{\frac{1}{2} (1+z^{-1})} \cdot \frac{1}{\frac{(1-z^{-1})}{\frac{1}{2} (1+z^{-1})} + a}$$

$$= \frac{b}{\frac{1}{2} (1+z^{-1})} \cdot \frac{2}{\frac{(1-z^{-1})}{1+z^{-1}} + a} = \frac{b}{\frac{1}{2} (1+z^{-1})} \cdot \frac{2}{\frac{1-z^{-1}}{1+z^{-1}} + a}$$

from eqn (9) and eqn (iii), — (iii)

$$H(z) = H_d(s) \Big|_{s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)}$$

Frequency mapping.

$$\text{As, } s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) = \frac{2}{T} \left(\frac{z-1}{z+1} \right)$$

where,

$$z = re^{j\omega} \quad \text{for simplicity, } r=1,$$

$$\therefore z = e^{j\omega} = \cos \omega + j \sin \omega$$

Then,

$$s = \frac{2}{T} \left(\frac{e^{j\omega} - 1}{e^{j\omega} + 1} \right)$$

$$= \frac{2}{T} \left(\frac{\cos \omega + j \sin \omega - 1}{\cos \omega + j \sin \omega + 1} \right)$$

$$= \frac{2}{T} \left[\frac{\cos \omega + j \sin \omega - 1}{\cos \omega + j \sin \omega + 1} \times \frac{\cos \omega - j \sin \omega + 1}{\cos \omega - j \sin \omega + 1} \right]$$

$$= \frac{2}{T} \left[\frac{(\cos \omega - 1) + j \sin \omega}{(\cos \omega + 1) + j \sin \omega} \times \frac{(\cos \omega + 1) - j \sin \omega}{(\cos \omega + 1) - j \sin \omega} \right]$$

$$= \frac{2}{T} \frac{(\cos \omega - 1)(\cos \omega + 1) - j \sin \omega \cos \omega + j \sin \omega + j \sin \omega + \sin^2 \omega}{(\cos \omega + 1)^2 + \sin^2 \omega}$$

$$= \frac{2}{T} \left[\frac{\cos^2 \omega - 1 + 2j \sin \omega + \sin^2 \omega}{\cos^2 \omega + 2 \cos \omega + 1 + \sin^2 \omega} \right]$$

$$= \frac{2}{T} \left[\frac{j \sin \omega}{2(\cos \omega + 1)} \right] \left[\frac{\sin^2 \omega + \cos^2 \omega}{\cos \omega + 1} \right]$$

$$= \frac{2}{T} \left[\frac{j \sin \omega}{\cos \omega + 1} \right]$$

$$= \frac{2}{T} \left[\frac{j \cdot 2 \cos \omega/2 \sin \omega/2}{\cos^2 \omega/2 + \sin^2 \omega/2 + \cos \omega/2 - \sin^2 \omega/2} \right]$$

$$= \frac{2}{T} \left[\frac{j \sin \omega/2 \cos \omega/2}{\cos \omega/2} \right]$$

$$= \frac{2}{T} \left[\frac{j \sin \omega/2}{\cos \omega/2} \right]$$

$$= j 2/T \tan(\omega/2)$$

$$\text{As, } s = \sigma + j\Omega$$

$$\Omega = 2/T \tan(\omega/2) \quad \#$$

IIR filter Design method by approximation
(Butterworth filter Design)

Designing steps:

Step-1] Obtain analog frequency from given digital filter specification.

$$\hookrightarrow \Omega = \frac{\omega}{T} \text{ using ITM}$$

$$\hookrightarrow \Omega = \frac{2}{T} \tan(\omega/2) \text{ using BLT}$$

Step-2] Obtain the order of filter (filter length)

$$N = \log_{10} \left[\frac{10^{0.1 A_s} - 1}{10^{0.1 A_p} - 1} \right]$$

if A_p and A_s are in dB (decibel).

$$2 \log_{10} \left(\frac{\Omega_s}{\Omega_p} \right)$$

or,

$$N = \log_{10} \left[\frac{1/A_s^2 - 1}{1/A_p^2 - 1} \right]$$

if A_s and A_p are not in dB.

$$2 \log_{10} \left(\frac{\Omega_s}{\Omega_p} \right)$$

Step-3] Obtain the cut off frequency (Ω_c).

i.e. $\Omega_c = \frac{\Omega_p}{\left[\frac{1}{A_p^2} - 1 \right]^{1/2N}}$; if A_p is not in dB.

or, $\Omega_c = \frac{\Omega_p}{\left[10^{0.1 A_p} - 1 \right]^{1/2N}}$; if A_p in dB.

Step-4] Obtain the poles locations.

$P_i = \pm \Omega_c e^{j(N+2i+1)\pi/2N}$; where $i = 0, 1, 2, \dots, N-1$

Step-5] Obtain the analog transformation.

$H_a(s) = \frac{N}{(s - P_0)(s - P_1) \dots}$

Step-6] Finally obtain the digital filter.

$H(z)$ using IIM or BLT.

Where, Ω_p = Passband edge frequency.

Ω_s = stopband edge frequency.

A_p = Passband attenuation.

A_s = stopband attenuation.

Ω_c = cut off frequency.

In analog terms.

ω_p = Passband edge frequency.

ω_s = Stop " "

ω_c = cut off " "

In digital terms.

$H(e^{j\omega}) \Rightarrow$ Discrete time fourier transform of $h(n)$

$H(z) \Rightarrow$ Digital transfer function.

$H_a(s) \Rightarrow$ Analog " "

* for Digital filter:

$|H(e^{j\omega})| \Rightarrow$ magnitude.

* for Passband:

$0 \leq |H(e^{j\omega})| \leq A_p$; for $0 \leq \omega \leq \omega_p$

* for Stopband:-

$|H(e^{j\omega})| \leq A_s$; for $\omega_s \leq \omega \leq \pi$.

calculator. $\rightarrow$ π for radian
used. otherwise degree

Q.7 Design a digital low pass ^{filter} filter to meet the following specifications by using Bilinear transformation method.

$$|H(e^{j\omega})| \leq 0.8 \quad |\omega| \leq 0.25\pi$$

$$0.25, \quad 0.45\pi < \omega < \pi$$

$\Rightarrow$ From given specification,

passband edge freq. (ω_p) = 0.25π .

passband attenuation (A_p) = 0.8

stopband edge freq. (ω_s) = 0.45π .

" attenuation (A_s) = 0.25

By Bilinear transformation method, we have,

$$\text{Analog freq. } (s_p) = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right)$$

At $T=1$, $\frac{2}{T} \tan\left(\frac{0.25\pi}{2}\right) = 0.8284$

again, $s_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right)$

$$= \frac{2}{T} \tan\left(\frac{0.45\pi}{2}\right)$$

At $T=1$, $= 2 \tan\left(\frac{0.45\pi}{2}\right) = 1.708$

step-2.

$$N = \lg \left[\frac{\frac{1}{A_s^2} - 1}{\frac{1}{A_p^2} - 1} \right]$$

, $N = \text{filter order}$

$$2 \lg_{10} \left(\frac{s_s}{s_p} \right)$$

= 2.26 (round up always take not float value only integer)

= 3

step-3

$\therefore s_c = s_p$

$$\left[\frac{1}{A_p^2} - 1 \right]^{\frac{1}{2N}}$$

= 0.911

step-4

$$P_i = + s_c e^{j(N+2i+\pi)/2N}$$

When $i=0$,

$P_0 = + 0.911 e^{j\pi/3}$

$$= + 0.911 \left[\cos \frac{2\pi}{3} + j \sin \frac{2\pi}{3} \right]$$

$$= + 0.911 [-0.5 + j 0.866]$$

$$\therefore p_0 = (-0.5 + j 0.866) s_c / (0.5 - j 0.866) s_c$$

(Take negative only)

$$p_0 = (-0.5 + j 0.866) s_c$$

i.e poles must lie in -ve s-plane.

When, $q = 1$.

$$p_1 = + s_c e^{j\pi}$$

$$= + s_c [\cos \pi + j \sin \pi]$$

$$= + s_c [-1]$$

$$= -s_c \quad \text{i.e. } p_1 = s_c / -s_c$$

$$\therefore p_1 = -s_c$$

When, $q = 2$.

$$p_2 = + s_c e^{j\frac{4\pi}{3}} = + s_c e^{j\frac{4\pi}{3}}$$

$$= + s_c \left[\cos \frac{4\pi}{3} + j \sin \frac{4\pi}{3} \right]$$

=

or,

Since, $N=3$; it must have 3-poles p_0, p_1, p_2 . Since p_1 has only real part.

$$\text{So, } p_2 = \bar{p}_0 \quad \text{So, } p_2 = (-0.5 - j 0.866) s_c$$

Step-5 The analog transfer function is,

$$H_a(s) = \frac{s_c^N}{[s_c(-0.5 + j 0.866)] [s_c + s_c] [s_c(-0.5 - j 0.866)]}$$

$$= \frac{s_c^3}{\{ (s_c + 0.5 s_c)^2 - (j 0.866 s_c)^2 \} (s_c + s_c)}$$

$$= \frac{s_c^3}{\{ 6^2 + 54s_c + 0.25s_c^2 + 0.86s_c^2 \} (s_c + s_c)}$$

$$= \frac{0.756}{(s^2 + 0.911s + 0.829)(s + 0.911)}$$

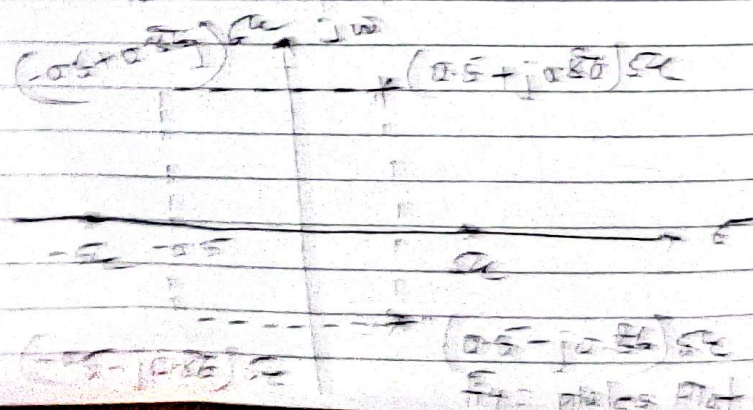
Step-6] Digital transfer function is,

$$H(z) = H(s) \left[s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) \right]$$

$$H(z) = \frac{0.756}{\left[\left(\frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) \right)^2 + 0.911 \left(\frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) \right) + 0.824 \left(\frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) + 0.911 \right) \right]}$$

above question.

* Plot the location of poles,



8] Design the digital low pass ~~with~~ Butterworth filter for the given specification.

Passband edge freq. (ω_p) = 1 kHz

" Attenuation (A_p) = 1 dB

Stop band edge freq. (ω_s) = 1.5 kHz

" " Attenuation (A_s) = 10 dB

Sampling frequency, (F_s) = 50 kHz

$$\text{Time period, } T = \frac{1}{F_s} = \frac{1}{50 \times 10^3} = 0.2 \times 10^{-4} \text{ sec.}$$

Now, the normalized frequencies are:-

$$\omega_p = 2\pi \left(\frac{f_p}{F_s} \right) = 2\pi \left(\frac{1}{50} \right) = 0.04\pi$$

$$\omega_s = 2\pi \left(\frac{f_s}{F_s} \right) = 2\pi \left(\frac{1.5}{50} \right) = 0.4\pi$$

Now, Analog frequencies are (using BLT)

$$\Omega_p = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right) = \frac{2}{0.2 \times 10^{-4}} \tan\left(\frac{0.04\pi}{2}\right) = 6283.185$$

$$\Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right) = \frac{2}{0.2 \times 10^{-4}} \tan\left(\frac{0.4\pi}{2}\right) = 72654.352$$

Step-2:

$$N = \log_{10} \left[\frac{10^{0.1 A_g - L}}{10^{0.1 A_p - L}} \right]$$

$$= 2 \log_{10} \left(\frac{S_{25}}{S_{cp}} \right)$$

$$= \log_{10} \left[\frac{10^{0.1 \times 10 - 1}}{10^{0.1 \times 1 - 1}} \right]$$

$$= 2 \log_{10} \left(\frac{72654.252}{6291.466} \right)$$

$$= 0.725$$

$\therefore N = 1$ [take round of value in integer form].

Step-3:

$$S_c = \left[\frac{S_p}{\left(10^{0.1 A_p - L} \right)^{1/N}} \right]$$

$$= \frac{6291.466}{\left(10^{0.1 \times 1 - 1} \right)^{1/1}} = 12364.157$$

Step-4 $N=1$

$$p_i = \pm S_c e^{j(N+2i+1)\pi/2N}$$

when $i=0$

$$p_0 = \pm S_c e^{j(1+2 \times 0 + 1)\pi/2}$$

$$= \pm S_c e^{j\pi}$$

$$p_0 = -S_c / S_c$$

$$\therefore p_0 = -S_c$$

Step-5

$$h_a(s) = \frac{S_c^N}{(s - p_0)}$$

$$= \frac{S_c}{(s + S_c)}$$

$$= \frac{12364.157}{(s + 12364.157)}$$

Step-6

Q. digital transfer function is,

$$h(z) = h(s) \Big|_{s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)}$$

$$= \left[\frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) + 12364.157 \right]$$

#.

Butterworth filter

Q.7 Design second order butterworth filter for the three dB (3dB) frequency of $\pi/2$.
sampling freq. (f_s) = 10 kHz

⇒ Given.

$$N = 2$$

$$\text{cut off frequency } \omega_c = \pi/2$$

~~3dB~~
Sampling freq. (f_s) = 10 kHz

$$\text{Time period } T = \frac{1}{f_s} = \frac{1}{10 \times 10^3}$$

$$= 0.1 \times 10^{-3} \text{ sec.}$$

By using BTL,

$$\sigma_c = \frac{2}{T} \tan \left(\frac{\omega_c}{2} \right)$$

$$= \frac{2}{0.1 \times 10^{-3}} \tan \left(\frac{\pi/2}{2} \right)$$

$$= 0.02 \times 0.0137 \cdot 20 \times 10^3$$

$$\sigma = 20 \times 10^3$$

#.

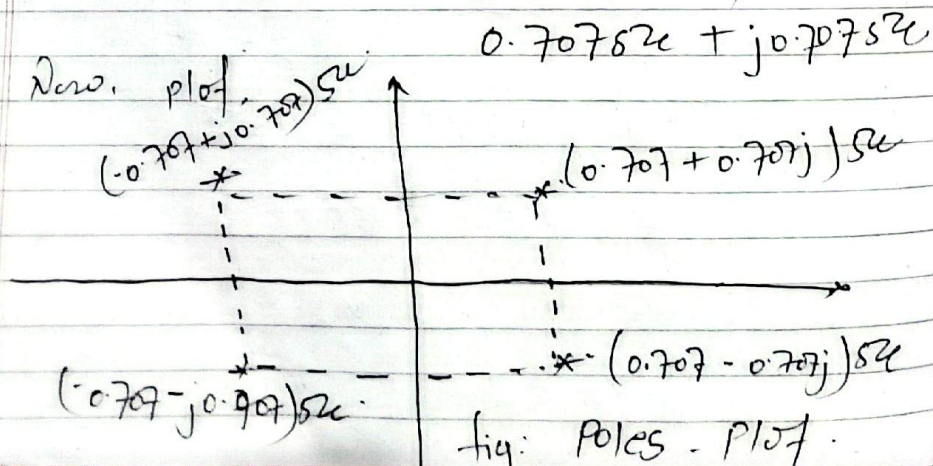
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$$\begin{aligned}
 p_i^0 &= \frac{+}{-} \omega_c e^{j(N+2i+1)\pi/2N} \\
 &= \frac{+}{-} \frac{\omega_c}{20 \times 10^3} e^{j(2+2i+1)\pi/2 \times 2} \\
 &= \frac{+}{-} \omega_c e^{j(3+2i)\pi/4}
 \end{aligned}$$

$i = 0$

$$\begin{aligned}
 p_0 &= \frac{+}{-} \omega_c e^{j(3\pi/4)} \\
 &= \frac{+}{-} \omega_c \left[\cos 3\pi/4 + j \sin 3\pi/4 \right] \\
 &= \frac{+}{-} \omega_c [0.707 + j 0.707] \\
 &= \frac{-}{+} 0.707 \omega_c + j 0.707 \omega_c / 0.707 \omega_c - j 0.707 \omega_c
 \end{aligned}$$

$P_1 = P_0^* = -0.707 \omega_c - 0.707j \omega_c /$



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Transformation function (Analog Transfer func):

$$H(s) = \frac{\omega_c^N}{(s-p_0)(s-p_1)}$$

$$= \frac{\omega_c^2}{s^2}$$

$$[s - (-0.707 + j0.707)\omega_c][s - (-0.707 - j0.707)\omega_c]$$

$$= \frac{\omega_c^2}{s^2}$$

$$[(s + 0.707\omega_c)^2 + (0.707\omega_c)^2]$$

$$= \frac{(20 \times 10^3)^2}{(s + 0.707\omega_c)^2 + (0.707\omega_c)^2}$$

Now, By BLT,

$$H(z) = H(s) \Big|_{s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)}$$

$$= \frac{C}{s_2} \left[\frac{2}{1+z^{-1}} + 0.707s_2 \right]^2 + (0.707s_2)^2$$

Put the value s_2 and T ,
to get final solution.

||

$$H(z) = \frac{(20000)^2}{\left[\frac{2}{0.1 \times 10^{-3}} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) + 0.707 \times 20000 \right]^2 + (0.707 \times 20000)^2}$$

$$= \frac{(20000)^2}{(20000)^2 \left[\left(\frac{1-z^{-1}}{1+z^{-1}} \right) + 0.707 \right]^2 + (0.707)^2} \quad \text{||}$$

By ZOH (Impulse Invariance method)

$$N = 2, \quad s_2 = \frac{\omega}{T} = \frac{\pi}{0.1}, \quad T = \frac{1}{f_s}$$

Poles,

$$P_0 = -s_2(-0.707 + j0.707)$$

$$P_1 = -s_2(-0.707 - j0.707)$$

Analog transfer function, $H_a(s)$

$$H_a(s) = \frac{84}{(s-P_0)(s-P_1)}$$

↓
do

infinite DTFT → Speech Signal processing
→ Seismograms
→ Image processing

Discrete-Time Fourier Transform

(finite) DFT (Discrete Fourier Transform)

→ The discrete time domain sequence $x(n)$ can be converted into its frequency domain by using discrete Fourier transform where the ~~square~~ sequence must be non-periodic.

→ The DFT is a finite version of DTFT, where,

$$x(n) \xrightarrow{\text{DFT}} X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

and IDFT (Inverse DFT) is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

Eqn (i) & (ii) are called DFT pair.

Q. Compute the four point (DFT) of the sequence $x(n) = \{1, 2, 1, 3\}$

Given,

$$x(n) = \{1, 2, 1, 3\}$$

$$\therefore N = 4$$

Note: $e^{-j\pi/2} = \cos \pi/2 - j \sin \pi/2 = -j$
 $e^{j\pi} = \cos \pi - j \sin \pi = -1$
 $e^{-j3\pi/2} = \cos 3\pi/2 - j \sin 3\pi/2 = j$

By definition of DFT, we have,

$$x(n) \xrightarrow{\text{DFT}} X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{4-1} x(n) e^{-j2\pi kn/4}$$

$$= \sum_{n=0}^3 x(n) e^{-j\pi kn/2}$$

$$= x(0) + x(1) e^{-j\pi k/2} + x(2) e^{-j\pi k} + x(3) e^{-j3\pi k/2}$$

$$\therefore X(k) = 1 + 2e^{-j\pi k/2} + e^{-j\pi k} + 3e^{-j3\pi k/2}$$

When, $k=0$; $X(0) = 1 + 2 + 1 + 3 = 7$

$$k=1; X(1) = 1 + 2e^{-j\pi/2} + e^{j\pi} + 2e^{-j3\pi/2}$$

$$= 1 - 2j - 1 + 2j = 0$$

$$k=2; X(2) = 1 + 2(-1) + (1) + 2(1)$$

$$= -2$$

$$k=3; X(3) = 1 + 2j + (-1) - 2j$$

$$= 0$$

$$\therefore x[k] = \{6, 0, -2, 0\} \#$$

Q7) $x(n) = \frac{y(n) - y(n-4)}{4}$; for $0 \leq n \leq 3$ $y[n-4] = 0$ $n \geq 4$

$$\therefore x(n) = \{2, 1, 1, 2\}$$

$N = 4$

Q8) Compute the 4 point DFT of the sequence $x(n)$ obtained by sampling $x_a(t) = 3 \sin 200\pi t$ at 800 Hz.

Given, Sampling freq. (f_s) = 800 Hz.

By sampling,

$$x(n) = x_a(t) \Big|_{t=nT_s} \quad x(n) = x_a(t) \Big|_{t=nT_s}$$

$$= 3 \sin(200\pi(nT_s))$$

$$= 3 \sin 200\pi n \frac{1}{800}$$

$$= 3 \sin(200\pi n \frac{1}{800})$$

$$= 3 \sin\left(\frac{\pi n}{4}\right)$$

$$T_s = \frac{1}{f_s} = \frac{1}{800}$$

Let, $n=0, 1, 2, 3$ for 4 points, then,

$$x(n) = \{0, 3/\sqrt{2}, 3, 3\sqrt{2}/2\}$$

By defⁿ of DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad \text{--- (1)}$$

$$= \sum_{n=0}^{4-1} x(n) e^{-j2\pi kn/4}$$

$$= \sum_{n=0}^3 x(n) e^{-j\pi kn/2}$$

$$= 0 + x(1) e^{-j\pi k/2} + x(2) e^{-j\pi k} + x(3) e^{-j3\pi k/2}$$

$$= 0 + \frac{3}{\sqrt{2}} e^{-j\pi k/2} + 3 e^{-j\pi k} + \frac{3}{\sqrt{2}} e^{-j3\pi k/2}$$

When,

$$k=0; \quad X(0) = 0 + \frac{3}{\sqrt{2}} + 3 + \frac{3}{\sqrt{2}} = 3 + 3\sqrt{2}$$

$$= 4 + \frac{6}{\sqrt{2}} = 8.24$$

$$= 7.24$$

Date _____
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$$k=1; \quad x(1) = \frac{3}{\sqrt{2}} e^{-j\pi/2} + 3 \cdot e^{-j\pi} + \frac{3}{\sqrt{2}} e^{-j3\pi/2}$$

$$= \frac{3}{\sqrt{2}} (j) + 3 \cdot (-1) + \frac{3}{\sqrt{2}} (-j)$$

$$= -3$$

$$k=2; \quad x(2) = \frac{3}{\sqrt{2}} (-1) + 3 \cdot 1 + \frac{3}{\sqrt{2}} (-1)$$

$$= 3 - \frac{3\sqrt{2}}{\sqrt{2}} = -1.24$$

$$k=3; \quad x(3) = \frac{3}{\sqrt{2}} (j) + 3 \cdot (-1) - \frac{3}{\sqrt{2}} j$$

$$= -3$$

So, $x(k) = \{ 1.24, -3, -1.24, -3 \}$

#. Properties of DFT

Q. Linearity property:-
It states that if $x_1(n) \xrightarrow[\text{DFT}]{N} X_1(k)$
and $x_2(n) \xrightarrow[\text{DFT}]{N} X_2(k)$

Then

$$x_3(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) \xrightarrow[\text{DFT}]{N} X_3(k)$$

$$= \alpha_1 X_1(k) + \alpha_2 X_2(k)$$

Where α_1 and α_2 are arbitrary constant.

Proof: We know,

$$\text{DFT} \{ x(n) \} = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$\therefore X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N} \quad \text{--- (i)}$$

$$\therefore X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N} \quad \text{--- (ii)}$$

Now, the linear combination of time domain sequence is.

$$x_3(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n)$$

$\therefore$ The corresponding DFT is.

$$\begin{aligned}
 X_3(k) &= \sum_{n=0}^{N-1} x_3(n) \cdot e^{-j2\pi kn/N} \\
 &= \sum_{n=0}^{N-1} \{ \alpha_1 x_1(n) + \alpha_2 x_2(n) \} e^{-j2\pi kn/N} \\
 &= \alpha_1 \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N} + \alpha_2 \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N} \\
 &= \alpha_1 X_1(k) + \alpha_2 X_2(k) \quad \text{--- (iii)}
 \end{aligned}$$

from eqn (i), (ii) and (iii), we get,

$$X_3(k) = \alpha_1 X_1(k) + \alpha_2 X_2(k) \quad \text{proved.}$$

Q4. Time shifting property:
It states that if $x(n) \xrightarrow{\text{DFT}} X(k)$
Then $x(n-n_0) \xrightarrow{\text{DFT}} e^{-j2\pi kn_0/N} X(k)$
Where ' n_0 ' is the time quantity
constant
proof, we have,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N}$$

$$\therefore \text{DFT}\{x(n-n_0)\} = \sum_{n=0}^{N-1} x(n-n_0) \cdot e^{-j2\pi kn/N}$$

let $n-n_0 = m$; As $n \rightarrow 0$ then $m = -n_0$
 $n = m+n_0$; $n \rightarrow N-1$ then $m = N-1-n_0$

$$\begin{aligned}
 \therefore \text{DFT}\{x(n-n_0)\} &= \sum_{m=-n_0}^{N-1-n_0} x(m) \cdot e^{-j2\pi k(m+n_0)/N} \\
 \therefore \text{DFT}\{x(n-n_0)\} &= \left(\sum_{m=0}^{N-1} x(m) \cdot e^{-j2\pi km/N} \right) \cdot e^{-j2\pi kn_0/N} \\
 &= e^{-j2\pi kn_0/N} \cdot \sum_{m=0}^{N-1} x(m) \cdot e^{-j2\pi km/N}
 \end{aligned}$$

$$= e^{-j2\pi kn_0/N} \cdot X(k)$$

hence proved.

$$\begin{aligned}
 \therefore \text{DFT}\{x(n-n_0)\} &= X(k) = \sum_{n=0}^{N-1} x(n-n_0) e^{-j2\pi kn/N} \\
 &= e^{-j2\pi kn_0/N} X(k)
 \end{aligned}$$

③ Frequency shifting property.

It states that if $x(n) \xrightarrow{\text{DFT}} X(k)$

Then $\underline{e^{j2\pi k_0 n/N} \cdot x(n)} \xrightarrow{\text{DFT}} \underline{X(k - k_0)}$

Proof: We know,

$$\text{DFT} \{ x(n) \} = X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N}$$

$$\therefore \text{DFT} \{ \underline{e^{j2\pi k_0 n/N} \cdot x(n)} \} = \sum_{n=0}^{N-1} \left(\underline{e^{j2\pi k_0 n/N} \cdot x(n)} \right)$$

$$= \sum_{n=0}^{N-1} x(n) \cdot \underline{e^{-j2\pi (k-k_0)n/N}}$$

$$= X(k - k_0)$$

proved

$$\therefore e^{j2\pi k_0 n/N} \cdot x(n) \xrightarrow{\text{DFT}} X(k - k_0)$$

S. m. 10.

④ Circular Convolution.

It states if $x_1(n) \xrightarrow{\text{DFT}} X_1(k)$

and $x_2(n) \xrightarrow{\text{DFT}} X_2(k)$

Then,

$$x_3(n) = x_1(n) \textcircled{N} x_2(n) \xrightarrow{\text{DFT}} X_3(k) = X_1(k) \cdot X_2(k)$$

$$= X_1(n) \textcircled{N} x_2(n)$$

Where, $\textcircled{N} \Rightarrow$ Symboling Circular Convolution.

Mathematically,

$$x_1(n) \textcircled{N} x_2(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2(n-m)$$

Proof: We know, by defⁿ of DFT and IDFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}; \text{ By DFT}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{+j2\pi kn/N}; \text{ By IDFT}$$

Now, taking IDFT of $X_3(k)$ we get,

$$x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} \underline{X_3(k) \cdot e^{j2\pi kn/N}}$$

since, $z = (\cos 2\pi p + j \sin 2\pi p)$

$$= \frac{1}{N} \sum_{k=0}^{N-1} x_1(k) \cdot \underbrace{x_2(k)}_z \cdot e^{j2\pi k p / N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{m=0}^{N-1} x_1(m) e^{-j2\pi m k / N} \right) \left(\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi l k / N} \right) e^{j2\pi k p / N}$$

Now, interchanging the order of summation

$$x(n) = \sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l) \cdot \frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi k (n-m-l) / N} \quad \text{--- (1)}$$

$$\text{Let } \sum_{k=0}^{N-1} e^{j2\pi k (n-m-l) / N} = \sum_{k=0}^{N-1} a^k$$

$$\text{Where, } a = e^{j2\pi (n-m-l) / N}$$

(Case-1) When $n-m-l$ is multiple of N

i.e. $n-m-l = \pm pN$

$$\therefore a = e^{j2\pi (\pm pN) / N} = e^{\pm j2\pi p} = 1 \quad \text{[above]}$$

$$\therefore \sum_{k=0}^{N-1} a^k = \sum_{k=0}^{N-1} 1^k = \sum_{k=0}^{N-1} 1 = N$$

(Case-2) When $n-m-l$ is not a multiple of N
i.e. $n-m-l \neq \pm pN$

$$\text{In this case, } \sum_{k=0}^{N-1} a^k = 1 + a + a^2 + \dots + a^{N-1} = \frac{1-a^N}{1-a} \quad \text{[sum of finite term geometric series]}$$

$$\therefore a^N = (e^{j2\pi (n-m-l) / N})^N$$

$$= e^{j2\pi (n-m-l)} = 1$$

$$\therefore \sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a} = \frac{1-1}{1-a} = 0$$

$$\therefore \sum_{k=0}^{N-1} e^{j2\pi k (n-m-l) / N} = \sum_{k=0}^{N-1} a^k = N \quad ; \text{ for } n-m-l = \pm pN$$

Hence eqⁿ (1) can be written as,

$$x(n) = \sum_{m=0}^{N-1} x_1(m) \cdot \sum_{l=0}^{N-1} x_2(l) \cdot \frac{1}{N}$$

$$\text{if } n-m-l = \pm pN \\ \therefore l = (n-m \pm pN) = (n-m \bmod N) = (n-m) / N$$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2(n-m)$$

Which is the circular convolution.

Note: 2. class remaining two write.

Q. Find the DFT of the following
 $\{x_1(n) \text{ (N)} x_2(n)\}$

Where,

$$x_1(n) = \{1, 2, 1, 2\}$$

$$x_2(n) = \{2, 3, 4, 5\}$$

or. find $x_1(k)$ & $x_2(k)$ then
 $X(k) = x_1(k) \cdot x_2(k)$

$$\Rightarrow x_1(n) \text{ (N)} x_2(n) \xrightarrow[\text{DFT}]{N} x_1(k) \cdot x_2(k)$$

$$x_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^3 x_1(n) e^{-j\pi kn/2}$$

(N)

$$= x_1(0) + x_1(1) \cdot e^{-j\pi k/2} + x_1(2) \cdot e^{-j\pi k} + x_1(3) \cdot e^{-j3\pi k/2}$$

$$x_1(k) = 1 + 2 \cdot e^{-j\pi k/2} + e^{-j\pi k} + 2 \cdot e^{-j3\pi k/2}$$

$$\text{When } k=0; x_1(0) = 1 + 2 + 1 + 2 = 6$$

$$k=1; x_1(1) = 1 + 2 \cdot e^{-j\pi/2} + e^{-j\pi} + 2 \cdot e^{-j3\pi/2}$$

$$= 1 + 2(-j) + (-1) + 2j$$

$$k=2; x_1(2) = 1 + 2 \cdot e^{-j\pi} + e^{-j2\pi} + 2 \cdot e^{-j3\pi}$$

$$= 1 + 2(-1) + 1 + 2(-1)$$

$$= -2$$

$$k=3; x_1(3) = 1 + 2 \cdot e^{-j3\pi/2} + e^{-j3\pi} + 2 \cdot e^{-j9\pi/2}$$

$$= 1 + 2j + (-1) + 2(-j)$$

$$= 0$$

So, $x_1(k) = \{6, 0, -2, 0\}$

Again,

$$X_2(k) = 2 + 3 \cdot e^{j\pi k/2} + 4 \cdot e^{j\pi k} + 5 \cdot e^{j3\pi k/2}$$

When,

$$k=0, X_2(0) = 2 + 3 + 4 + 5 = 14$$

$$k=1, X_2(1) = 2 + 3 \cdot j + 4 \cdot (-1) + 5 \cdot j \\ = -2 + 2j$$

$$k=2, X_2(2) = 2 + 3 \cdot (-1) + 4 \cdot (1) + 5 \cdot (-1) \\ = -2$$

$$k=3, X_2(3) = 2 + 3j + 4 \cdot (-1) + 5 \cdot (-j) \\ = -2 + (-2j) \\ = -2 - 2j$$

So,

$$X_2(k) = \{14, -2+2j, -2, -2-2j\}$$

Now, By

$$X(k) = X_1(k) \cdot X_2(k)$$

$$= \{6, 0, -2, 0\} \cdot \{14, -2+2j, -2, -2-2j\}$$

$$= \{84, 0, 4, 0\}$$

#.

DFT as a linear transformation.
We know,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{j2\pi kn/N} \\ = \sum_{n=0}^{N-1} x(n) \cdot \omega_N^{kn}$$

Where, $\omega_N = e^{j2\pi/N}$ $\begin{cases} n=0 \text{ to } N-1 \\ k=0 \text{ to } N-1 \end{cases}$

Similarly, IDFT is,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j2\pi kn/N}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-jkn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$$

The $x(k)$, $X(k)$ and W_N can be written in matrix form as

$$\underline{X(k)} = X_N \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} \quad ; \quad \underline{x(n)} = x_N = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^1 & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)^2} \end{bmatrix} \quad N \times N$$

For $N=4$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \quad 4 \times 4$$

where $W_N = e^{-j2\pi/N}$
 $W_4 = e^{-j2\pi/4} = e^{-j\pi/2} = -j$

$$W_4^2 = (-j)^2 = -1$$

$$W_4^3 = -j = 1 \times (-j) \times (-1) \times (-j) = -1$$

$$W_4^4 = 1 = W_4^0 \quad W_4^5 = W_4^1$$

$$W_4^6 = -1 = W_4^2 \quad W_4^7 = W_4^3$$

$$W_4^8 = 1 = W_4^0 \quad W_4^9 = W_4^1$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad 4 \times 4$$

$$W_N = e^{-j2\pi/N}$$

$$\therefore X(k) = X_4 = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = W_4 \cdot x_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \times \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

eg. $x(n) = \{1, 2, 3, 4\}$ Then, 4 points DFT is,

$$X(k) = X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+2j-3-4j \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$X(k) = \{10, -2+2j, -2, -2-2j\}$$

Q.7 Compare the DFT of $\{4(n) - 4(n-5)\}$ (N) $h(n)$ where $h(n) = \{1, 2, 3, 4\}$

$$\Rightarrow \text{let } x(n) = \{4(n) - 4(n-5)\}$$

$$\therefore x(n) = \{1, 2, 3, 4\}$$

then

$$x(n) \otimes h(n) \xrightarrow[N]{\text{DFT}} X(k) \cdot H(k)$$

$$N=4$$

$$X_N = W_N X_N = W_4 \cdot x_4$$

$$X(k) = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1 \\ 1-j-1+j \\ 1-1+1-1 \\ 1+j-1-j \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Similarly,

$$H(k) = H_N = H_4 = W_N \cdot h_N$$

$$= W_4 \cdot h_4$$

$$H(k) = \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

4x1 4x4 4x1

$$= \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+2j-3-4j \end{bmatrix} \cdot X(k)$$

$$= \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

finally, $y(k) = X(k) \cdot H(k)$

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$X_N = W_N x_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

#. Twiddle factor for $M = 8 \div W_8$

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ 1 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ 1 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ 1 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ 1 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ 1 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix}$$

8x8

#. Properties of twiddle factors :-

① Periodicity :- $W_N^k = W_N^{k+N}$

Proof :- $W_N = e^{-j2\pi/N}$

$$W_N^k = (e^{-j2\pi/N})^k$$

Hence,

$$W_N^{k+N} = (e^{-j2\pi/N})^{k+N} = (e^{-j2\pi/N})^k \cdot (e^{-j2\pi/N})^N$$

$$= (e^{-j2\pi/N})^k \cdot e^{-j2\pi}$$

$$= W_N^k \cdot 1 \quad \left[\because e^{-j2\pi} = \cos 2\pi - j\sin 2\pi \right]$$

$2\pi = 1$

$$\left[\because w_N^{k+N} = w_N^k \right] \text{ proved.}$$

(ii) Symmetric property :-

$$w_N^{k+N/2} = -w_N^k$$

Proof We know, $w_N = e^{-j2\pi/N}$

$$\begin{aligned} \therefore w_N^{k+N/2} &= (e^{-j2\pi/N})^{k+N/2} \\ &= (e^{-j2\pi/N})^k \cdot (e^{-j2\pi/N})^{N/2} \\ &= w_N^k \cdot e^{-j\pi} \\ &= w_N^k \cdot (-1) \quad [\because e^{-j\pi} = \cos\pi - j\sin\pi] \\ &= -w_N^k \quad [\because \cos\pi = -1, \sin\pi = 0] \end{aligned}$$

$$\left[\because w_N^{k+N/2} = -w_N^k \right] \text{ Hence proved.}$$

(3) Duality Property :-

$$w_N^2 = w_{N/2}$$

$$\therefore w_N^2 = (e^{-j2\pi/N})^2$$

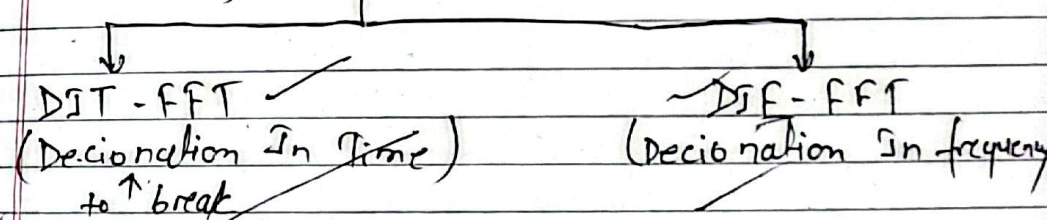
$$= e^{-j2\pi/N/2}$$

$$\left[\because w_N^2 = w_{N/2} \right]$$

$$\left[\text{i.e. } w_4^1 = w_8^2 \right]$$

#. FFT algorithm (Fast Fourier Transform).

DFT by FFT Algorithm.



Q] Why FFT is used even though DFT is already existed?

⇒ We know, by DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N}$$

∴ IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j2\pi kn/N}$$

For the N -point DFT computation will require

- ① N^2 multiplication
- ② $N(N-1)$ Addition

Due to this computational complexity of DFT the FFT is evolved

* Ex: 2:-

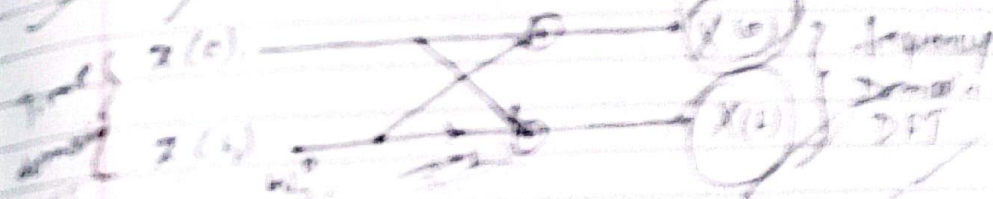
For the computation of DFT by FFT algorithm the value of N -point must be power of 2 i.e. $N = 2^L$. Here it is called radix-2 or base-2

* Types of FFT

① DIT-FFT:

- The time sequence $x(n)$ is broken into its even part and odd part.
- The even parts are divided by $N/2$ difference so as the odd parts.
- The FFT algorithm is represented by signal flow graph as shown in the figure

i) For $N=2$, $x(-1) = \frac{1}{2} x(0) + \frac{1}{2} x(1)$



by 2-point DIT-FFT algorithm (Butterfly Structure)

where,

$$X(0) = x(0) + x(1) \cdot W_N^0$$

$$X(1) = x(0) - x(1) \cdot W_N^1$$

ii) For $N=4$

$x(n) = \{x(0), x(1), x(2), x(3)\}$



fig: 4-point DST-FFT algorithm.

Where, $W_N = e^{-j2\pi/N}$

$$W_4 = e^{-j2\pi/4} = e^{-j\pi/2} = \cos \pi/2 - j \sin \pi/2$$

$$[W_4^1 = -j]$$

output of stage-1:

$$S_1(0) = x(0) + x(2)W_4^0$$

$$S_1(1) = x(0) - x(2)W_4^1$$

$$S_1(2) = x(1) + x(3)W_4^0$$

$$S_1(3) = x(1) - x(3)W_4^1$$

output of stage-2 or final stage.

$$X(0) = S_1(0) + S_1(2)W_4^0$$

$$X(1) = S_1(1) + S_1(3)W_4^1$$

$$X(2) = S_1(0) - S_1(2)W_4^0$$

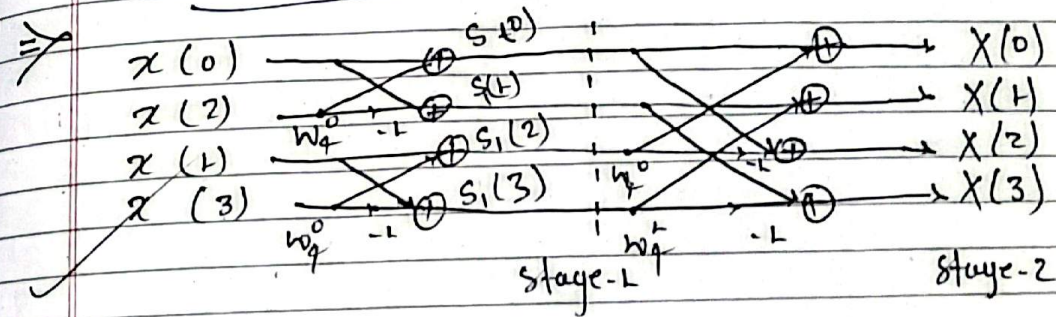
$$X(3) = S_1(1) - S_1(3)W_4^1$$

finally,

$$x(k) = \{X(0), X(1), X(2), X(3)\}$$

$W_4^0 = 1$ always.

Q1. Compute the four point DFT by FFT algorithm for $x(n) = \{1, 2, 3, 4\}$.



output of stage-1:

$$S_1(0) = x(0) + x(2)W_4^0 = 1 + 3 = 4$$

$$S_1(1) = x(0) - x(2)W_4^1 = 1 - 3 = -2$$

$$S_1(2) = x(1) + x(3)W_4^0 = 2 + 4 = 6$$

$$S_1(3) = x(1) - x(3)W_4^1 = 2 - 4 = -2$$

output of final stage :-

$$X(0) = S_1(0) + S_1(2)W_4^0 = 4 + 6 = 10$$

$$X(1) = S_1(1) + S_1(3)W_4^1 = -2 + 2j$$

$$X(2) = S_1(0) \cdot S_1(2) w_8^0 = -2$$

$$X(3) = S_1(1) - S_1(3) w_8^1 = -2 - 2j$$

finally $X(k) = \{x(0), x(1), x(2), x(3)\}$
 $= \{10, -2+2j, -2, -2-2j\}$

(iii) $N=8$; $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

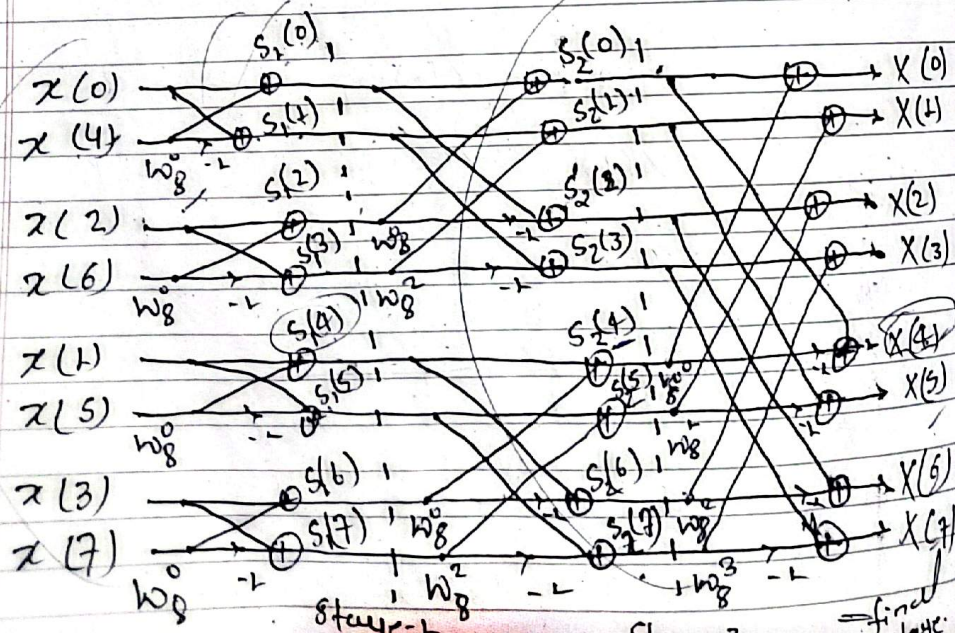


fig: 8-point DST-FFT algorithm.

Where,

$$w_8 = e^{-j2\pi/8} = e^{-j\pi/4} = \cos \pi/4 - j \sin \pi/4 = 0.707 - j0.707$$

$$w_8^1 = 0.707 - j0.707$$

$$w_8^2 = (e^{-j\pi/4})^2 = e^{-j\pi/2} = \cos \pi/2 - j \sin \pi/2 = (-j)$$

$$w_8^3 = w_8^2 \cdot w_8^1 = (-j) \cdot (0.707 - j0.707) = 0.707 - j0.707$$

output of stage-1:

$$\begin{aligned} S_1(0) &= x(0) + x(4) w_8^0 \\ S_1(1) &= x(0) - x(4) w_8^0 \\ S_1(2) &= x(2) + x(6) w_8^0 \\ S_1(3) &= x(2) - x(6) w_8^0 \\ S_1(4) &= x(1) + x(5) w_8^0 \\ S_1(5) &= x(1) - x(5) w_8^0 \\ S_1(6) &= x(3) + x(7) w_8^0 \\ S_1(7) &= x(3) - x(7) w_8^0 \end{aligned}$$

output of stage - 2:

$$S_2(0) = S_1(0) + S_1(2) w_8^0$$

$$S_2(1) = S_1(1) + S_1(3) w_8^2$$

$$S_2(2) = S_1(0) - S_1(2) w_8^0$$

$$S_2(3) = S_1(1) - S_1(3) w_8^2$$

$$S_2(4) = S_1(4) + S_1(6) w_8^0$$

$$S_2(5) = S_1(5) + S_1(7) w_8^2$$

$$S_2(6) = S_1(4) - S_1(6) w_8^0$$

$$S_2(7) = S_1(5) - S_1(7) w_8^2$$

output of final stage.

$$X(0) = S_2(0) + S_2(4) w_8^0$$

$$X(1) = S_2(1) + S_2(5) w_8^1$$

$$X(2) = S_2(2) + S_2(6) w_8^2$$

$$X(3) = S_2(3) + S_2(7) w_8^3$$

$$X(4) = S_2(0) - S_2(4) w_8^0$$

$$X(5) = S_2(1) - S_2(5) w_8^1$$

$$X(6) = S_2(2) - S_2(6) w_8^2$$

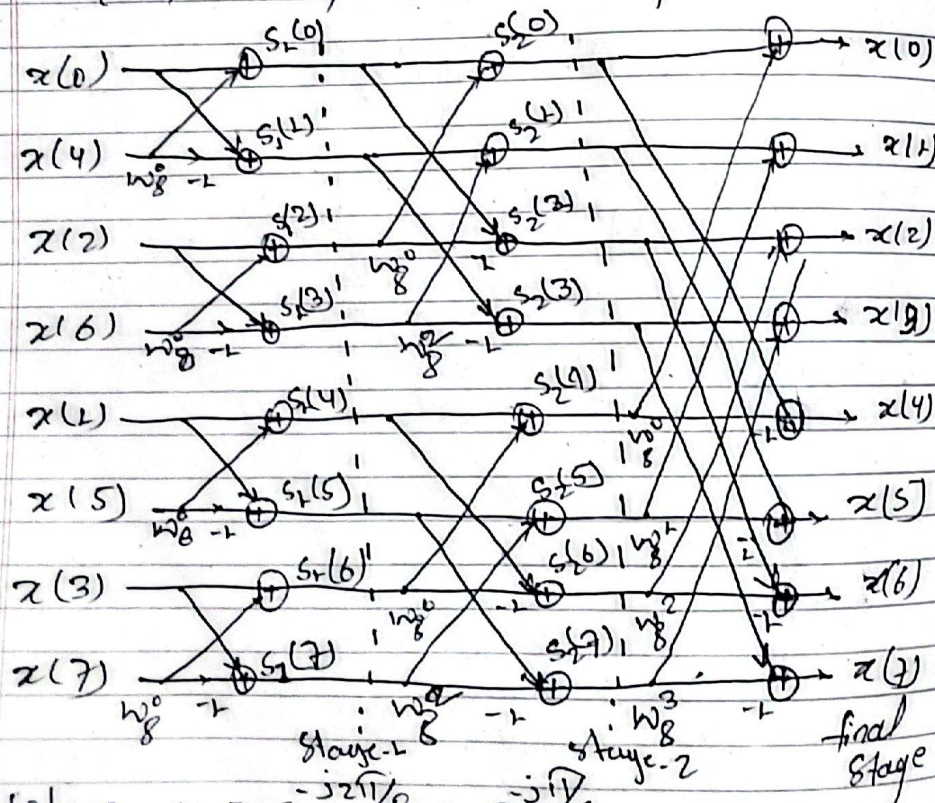
$$X(7) = S_2(3) - S_2(7) w_8^3$$

finally $x(k) = \{ \text{-----} \}$

Q. Compute the 8-point DFT by using DIT-FFT algorithm for the sequence.

$$x(n) = \{0.707, 1, 0.707, 0, -0.707, -1, -0.707, 0\}$$

⇒ We know, by DIT-FFT algorithm



Where $w_8 = e^{-j2\pi/8} = e^{-j\pi/4}$
 $= \cos \frac{\pi}{4} - j \sin \frac{\pi}{4}$

$$= 0.707 - j0.707$$

$$w_8^1 = 0.707 - j0.707$$

$$w_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$$

$$w_8^3 = w_8^2 \cdot w_8^1 = (-j)(0.707 - j0.707) = -j0.707 + 0.707 = 0.707 - j0.707$$

output stage-1:

$$\begin{aligned} S_1(0) &= x(0) + x(4) \cdot w_8^0 = 0.707 - 0.707 = 0 \\ S_1(1) &= x(0) - x(4) \cdot w_8^0 = 0.707 + 0.707 = 1.414 \\ S_1(2) &= x(2) + x(6) \cdot w_8^0 = 0.707 - 0.707 = 0 \\ S_1(3) &= x(2) - x(6) \cdot w_8^0 = 0.707 + 0.707 = 1.414 \\ S_1(4) &= x(1) + x(5) \cdot w_8^0 = 1 - 1 = 0 \\ S_1(5) &= x(1) - x(5) \cdot w_8^0 = 1 + 1 = 2 \\ S_1(6) &= x(3) + x(7) \cdot w_8^0 = 0 + 0 = 0 \\ S_1(7) &= x(3) - x(7) \cdot w_8^0 = 0 - 0 = 0 \end{aligned}$$

output of stage-2

$$S_2(0) = S_1(0) + S_1(2) w_8^0 = 0 + 0 = 0$$

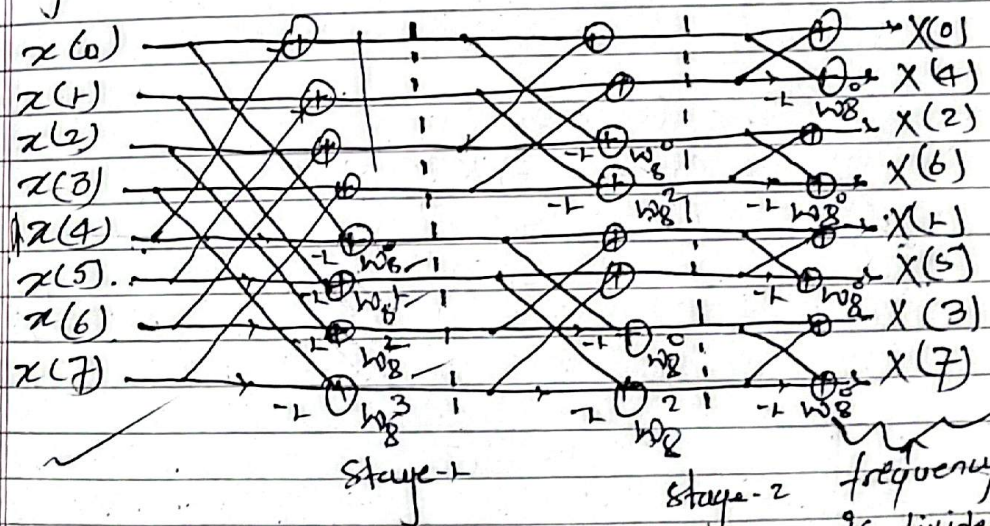
$$\begin{aligned}
 S_2(1) &= S_1(1) + S_1(3) w_8^0 = 1.414 + (-j1.414) \\
 S_2(2) &= S_1(0) - S_1(2) w_8^0 = 0 \\
 S_2(3) &= S_1(1) - S_1(3) w_8^2 = \\
 S_2(4) &= S_1(4) + S_1(6) w_8^0 = \\
 S_2(5) &= S_1(5) + S_1(7) w_8^2 = \\
 S_2(6) &= S_1(6) - S_1(6) w_8^0 = \\
 S_2(7) &= S_1(7) - S_1(7) w_8^2 =
 \end{aligned}$$

output of final stage.

$$\begin{aligned}
 X(0) &= S_2(0) + S_2(4) w_8^0 = \\
 X(1) &= S_2(1) + S_2(5) w_8^1 = \\
 X(2) &= S_2(2) + S_2(6) w_8^2 = \\
 X(3) &= S_2(3) + S_2(7) w_8^3 = \\
 X(4) &= S_2(0) - S_2(4) w_8^0 = \\
 X(5) &= S_2(1) - S_2(5) w_8^1 = \\
 X(6) &= S_2(2) - S_2(6) w_8^2 = \\
 X(7) &= S_2(3) - S_2(7) w_8^3 =
 \end{aligned}$$

DIF-FFT algorithm (Decimation In Frequency Fast Fourier Transform).

- Frequency domain is decimated into even parts and odd parts.
- Even parts and odd parts are separated by $(N/2)$ factors.
- The division of points are reverse in sequence from DFT-FFT for DIF-FFT.



frequency is divided by $(N/2)$

output of Stage-1

$$\begin{aligned} S_1(0) &= x(0) + x(4) \\ S_1(1) &= x(1) + x(5) \\ S_1(2) &= x(2) + x(6) \\ S_1(3) &= x(3) + x(7) \\ S_2(4) &= [x(0) - x(4)] w_8^0 \\ S_1(5) &= [x(1) - x(5)] w_8^1 \\ S_1(6) &= [x(2) - x(6)] w_8^2 \\ S_1(7) &= [x(3) - x(7)] w_8^3 \end{aligned}$$

output of Stage-2

$$\begin{aligned} S_2(0) &= S_1(0) + S_1(2) \\ S_2(1) &= S_1(1) + S_1(3) \\ S_2(2) &= [S_1(0) - S_1(2)] w_8^2 \\ S_2(3) &= [S_1(1) - S_1(3)] w_8^1 \\ S_2(4) &= S_1(4) + S_1(6) \\ S_2(5) &= S_1(5) + S_1(7) \\ S_2(6) &= [S_1(4) - S_1(6)] w_8^0 \\ S_2(7) &= [S_1(5) - S_1(7)] w_8^3 \end{aligned}$$

output of final stage

$$\begin{aligned} X(0) &= S_2(0) + S_2(4) \\ X(4) &= [S_2(0) - S_2(4)] w_8^0 \\ X(2) &= S_2(2) + S_2(3) \end{aligned}$$

$$\begin{aligned} X(6) &= [S_2(2) - S_2(3)] w_8^0 \\ X(1) &= S_2(4) + S_2(5) \\ X(5) &= [S_2(4) - S_2(5)] w_8^0 \\ X(3) &= S_2(6) + S_2(7) \\ X(7) &= [S_2(6) - S_2(7)] w_8^0 \end{aligned}$$

Inverse DFT by DIF-FFT algorithm :-
We know, by DFT:

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} x(n) w_N^{kn}, \text{ where } w_N = e^{-j2\pi/N} \end{aligned}$$

Also, by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) w_N^{-kn}, \text{ where } w_N = e^{j2\pi/N}$$

for inverse DFT by FFT algorithm

→ The frequency domain $X(k)$ is divided N .

→ The time domain sequence $x(n)$ is computed from $x(k)$.

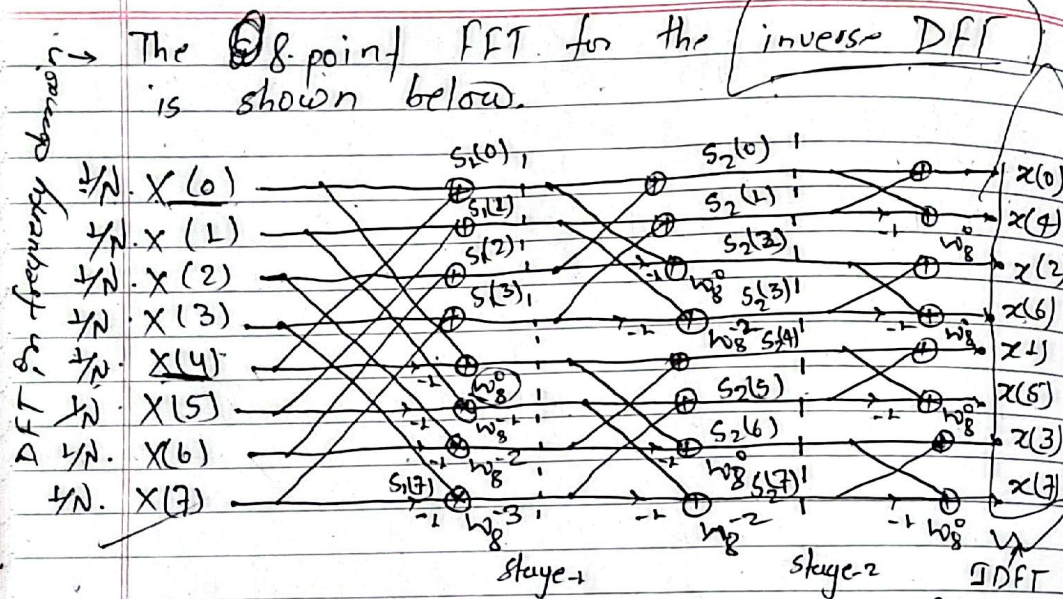


fig. 8-point FFT for IDFT

output of stage-1:

$$\begin{aligned}
 S_1(0) &= \frac{1}{4N} [x(0) + x(4)] \\
 S_1(1) &= \frac{1}{4N} [x(1) + x(5)] \\
 S_1(2) &= \frac{1}{4N} [x(2) + x(6)] \\
 S_1(3) &= \frac{1}{4N} [x(3) + x(7)] \\
 S_1(4) &= \frac{1}{4N} [x(0) - x(4)] w_8^0 \\
 S_1(5) &= \frac{1}{4N} [x(1) - x(5)] w_8^{-1} \\
 S_1(6) &= \frac{1}{4N} [x(2) - x(6)] w_8^{-2} \\
 S_1(7) &= \frac{1}{4N} [x(3) - x(7)] w_8^{-3}
 \end{aligned}$$

output of stage-2:

$$\begin{aligned}
 S_2(0) &= S_1(0) + S_1(2) \\
 S_2(1) &= S_1(1) + S_1(3) \\
 S_2(2) &= [S_1(0) - S_1(2)] w_8^0 \\
 S_2(3) &= [S_1(1) - S_1(3)] w_8^{-2} \\
 S_2(4) &= S_1(4) + S_1(6) \\
 S_2(5) &= S_1(5) + S_1(7) \\
 S_2(6) &= [S_1(4) - S_1(6)] w_8^0 \\
 S_2(7) &= [S_1(5) - S_1(7)] w_8^{-2}
 \end{aligned}$$

output of final stage-

$$\begin{aligned}
 x(0) &= S_2(0) + S_2(4) \\
 x(1) &= \cancel{S_2(0)} - \cancel{S_2(4)} w_8^0 [S_2(0) - S_2(4)] w_8^0 \\
 x(2) &= S_2(2) + S_2(6) \\
 x(3) &= [S_2(2) - S_2(6)] w_8^0 \\
 x(4) &= \cancel{S_2(0)} - \cancel{S_2(4)} S_2(4) + S_2(5) \\
 x(5) &= [S_2(4) - S_2(5)] w_8^0 \\
 x(6) &= S_2(6) + S_2(7) \\
 x(7) &= [S_2(6) - S_2(7)] w_8^0
 \end{aligned}$$

Where, $w_N = e^{-j2\pi/N}$

$$w_8 = e^{-j2\pi/8}$$

$$w_8^{-1} = e^{j\pi/4} = \cos \pi/4 + j \sin \pi/4$$

$$= 0.707 + j0.707$$

$$w_8^{-2} = e^{+j\pi/2} = \cos \pi/2 + j \sin \pi/2$$

$$= j$$

$$w_8^3 = w_8^{-2} \cdot w_8^{-1}$$

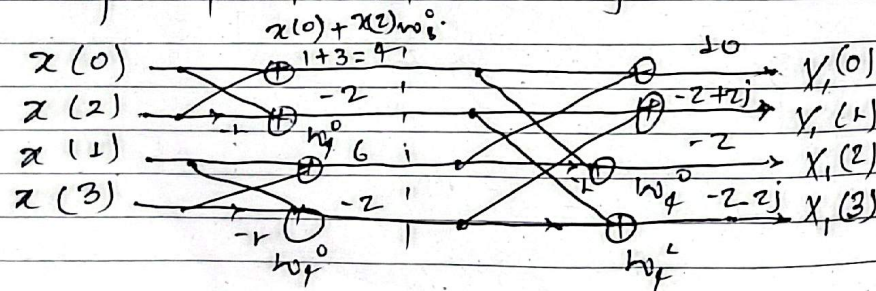
$$= -0.707 + j0.707$$

Q.7 Compute the Circular Convolution by using FFT algorithm, given by

$$x_1(n) = \{1, 2, 3, 4\}$$

$$x_2(n) = \{2, 1, 1, 2\}$$

⇒ Using 4-point DFT-FFT algorithm

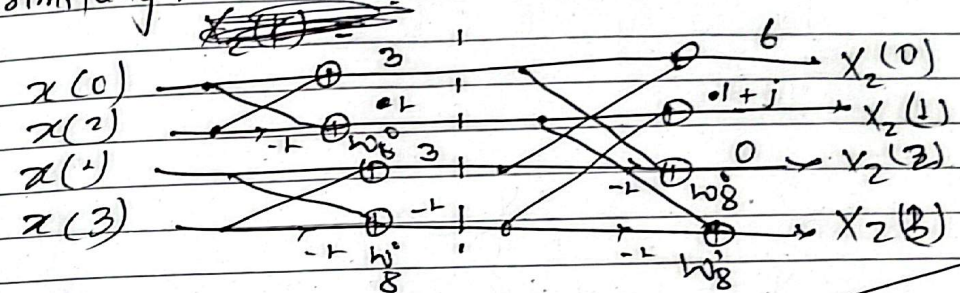


Where, $w_N = e^{-j2\pi/N} \Rightarrow w_4 = e^{-j2\pi/4} = 1$

$$w_4^1 = e^{-j\pi/2} = -j$$

$$\therefore x_1(k) = \{1, -2+j, -2, -2-j\}$$

Similarly,



$$x_2(k) = \{6, 1+j, 0, 1-j\}$$

Then,

use $X_3(k) = X_1(k) \cdot X_2(k)$

$$= \{60\}$$

Derivation of $H_d(n)$

Q.N.] Design the symmetrical FIR (low pass) filter which the direct response is expressed as

$$H_d(\omega) = \begin{cases} e^{-j\omega N} & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Now, taking the inverse Fourier transform, we get

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega N} \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega (N-n)} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega (N-n)}}{j(N-n)} \right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega_c (N-n)} - e^{-j\omega_c (N-n)}}{j(N-n)} \right]$$

$$\therefore h_d(n) = \frac{1}{\pi(N-n)} \left[\frac{e^{j\omega_c (N-n)} - e^{-j\omega_c (N-n)}}{2j} \right]$$

$$= \frac{1}{\pi(N-n)} \sin \omega_c (N-n) \left[\because \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \right]$$

$$\therefore h_d(n) = \frac{\sin \omega_c (N-n)}{\pi(N-n)} \quad \text{--- (1)}$$

$$\sin(-\theta) = -\sin\theta$$

where, ' τ ' is the delay (time delay)
for symmetric FIR filter,

$$h_d(n) = h_d[M-1-n]$$

eqn (i) can be written as,

$$\frac{\sin \omega_c (n-\tau)}{\pi (n-\tau)} = \frac{\sin \omega_c (M-1-n-\tau)}{\pi (M-1-n-\tau)} \quad \text{--- (ii)}$$

$$-(n-\tau) = M-1-n-\tau$$

$$\text{i.e. } 2\tau = M-1$$

$$\therefore \tau = \frac{M-1}{2} \quad \#$$

eqn (i) can be written as,

$$h_d(n) = \frac{\sin \omega_c (n - \frac{M-1}{2})}{\pi (n - \frac{M-1}{2})}$$

$$h_d(n) = \frac{\omega_c}{\pi} \quad ; \text{ for } n = \tau = \frac{M-1}{2}$$

$$\frac{\sin \omega_c (n - \frac{M-1}{2})}{\pi (n - \frac{M-1}{2})} \quad \text{for } n = \tau = \frac{M-1}{2}$$

Unit-6

FIR filter Design:

1. Rectangular window:

first, window method

Step-1] find ideal frequency response

Ex: ideal low pass filter.

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Step-2] find ideal impulse response, take inverse DFT:

$$h_d(n) = \frac{\sin(\omega_c(n-M))}{\pi(n-M)}$$

Where,

$$M = \frac{N-1}{2}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

Step-3] Apply window function.

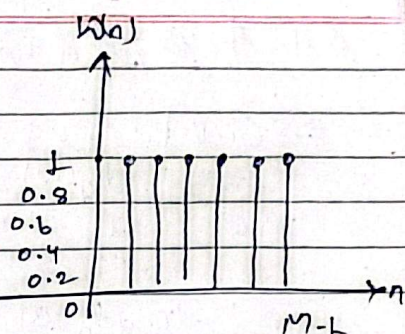
$$h(n) = h_d(n) \cdot w(n)$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

1. Rectangular window,

$$w(n) = 1$$

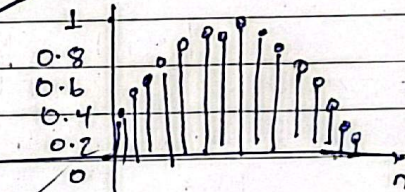
- Small transition width
- large ripple
- Worst Gibbs effect.



2. Hanning window,

$$w(n) = 0.5 - 0.5 \cos\left(\frac{2\pi n}{M-1}\right)$$

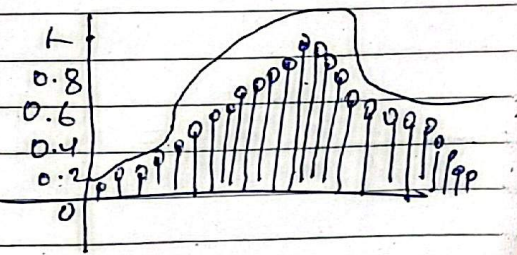
- less ripple
- wider transition.



3. Hamming window,

$$w(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{M-1}\right)$$

- Better stopband attenuation
- most commonly used.



Q7. Rectangular window,

↳ Given,

$$h_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

length of the filter $M = 7$

Cut off frequency $(\omega_c) = \pi$ radian/sample.

We know,

Unit Sample response is given by,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h_d(\omega) e^{j\omega n} d\omega \quad \text{--- (i)}$$

Here,

$$h_d(\omega) = \begin{cases} e^{j\omega\tau} & \text{for } -\pi \leq \omega \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

So, eqn (i) becomes,

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega\tau} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(\tau+n)} d\omega \end{aligned}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(\tau+n)}}{j(\tau+n)} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\pi(\tau+n)} - e^{-j\pi(\tau+n)}}{j(\tau+n)} \right]$$

$$= \frac{1}{\pi(\tau+n)} \left[\frac{e^{j\pi(\tau+n)} - e^{-j\pi(\tau+n)}}{2j} \right]$$

$$= \frac{\sin(\pi(\tau+n))}{\pi(\tau+n)} \quad \text{for } n \neq -\tau$$

When, $n = -\tau$, then,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega = \frac{1}{\pi}$$

So,

$$h_d(n) = \begin{cases} \frac{\sin(\pi(\tau+n))}{\pi(\tau+n)} & \text{for } n \neq -\tau \\ \frac{1}{\pi} & \text{for } n = -\tau \end{cases}$$

the is symmetric,

$$h(n) = h(M-1-n) \quad \text{--- (iii)}$$

we know,

$$h(n) = h_d(n) \cdot w(n) \quad \text{--- (iv)}$$

from eqn (ii) and (iii),

$$h_d(n) w(n) = h_d(M-1-n) w(n)$$

$$h_d(n) = h_d(M-1-n)$$

from eqn (ii) we get,

$$\frac{\sin(n\tau)}{\pi(n\tau)} = \frac{\sin(M-1-n\tau)}{\pi(M-1-n\tau)}$$

$$\tau = \frac{M-1}{2}$$

Again eqn (ii) becomes,

$$h_d(n) = \begin{cases} \frac{\sin\left(n - \frac{M-1}{2}\right)}{\pi\left(n - \frac{M-1}{2}\right)} & \text{for } n \neq \frac{M-1}{2} \\ \frac{1}{\pi} & \text{for } n = \frac{M-1}{2} \end{cases}$$

Here, $M=7$. So,

$$h_d(n) = \begin{cases} \frac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \frac{1}{\pi} & \text{for } n = 3 \end{cases} \quad \text{--- (v)}$$

By tabulation method, from eqn (v),

When (n)	$h_d(n)$ value of coefficient
0	0.015
1	0.1447
2	0.2678
3	0.3183
4	0.2678
5	0.1447
6	0.015

we know that, Rectangular window,

$$w(n) = \begin{cases} 1 & ; 0 \leq n \leq 6 \\ 0 & ; \text{otherwise.} \end{cases}$$

Now, required solution is,

$$h(n) = h_d(n) \cdot w(n) \quad \text{--- (vi)}$$

By tabulation method,

n	$h(n) = h_d(n) \cdot w(n)$
0	
1	✓
2	✓
3	✓
4	✓
5	✓
6	✓

Again, $h(n) = h(M-1-n)$
 $= h(7-1-n)$

$h(n) = h(6-n)$ So,

$h(0) = h(6)$

$h(1) = h(5)$

$h(2) = h(4)$

#1.

→ Hanning window,

Unit-6

Numerical:

1. 6.a]

Here,

Given,

3dB cut off frequency at 30 rad/sec

i.e. $\omega_c = 30 \text{ rad/sec}$

Sampling frequency (f_s) = 100 Hz

stopband Attenuation of 50 dB at $45\pi \text{ rad/sec}$

i.e. $A_s = 50 \text{ dB}$, $\omega_s = 45\pi \text{ rad/sec}$

we know,

$$\omega = 2\pi f \quad \text{--- (i)}$$

$$f = \frac{f}{f_s}$$

$$\omega = \frac{2\pi F}{f_s} \quad \text{--- (ii)}$$

$$\omega_c = 2\pi F$$

$$\text{So, } \omega = \frac{\omega_c}{f_s} \quad \text{--- (iii)}$$

$$\omega_1 = \frac{\omega_c}{f_s} = \frac{30}{100} = 0.3\pi \text{ rad/sec}$$

$$\omega_2 = \frac{\omega_s}{f_s} = \frac{45}{100} = 0.45\pi \text{ rad/sec}$$

Now, select Hamming window, because it provide ~~40~~ 50 dB of stopband attenuation.

So, find the filter order,

The main lobe of this window is $= \frac{8\pi}{M}$

$$\text{i.e. } k \left(\frac{2\pi}{M} \right) = \frac{8\pi}{M}$$

$$k = 4$$

The order of filter is,

$$N = K \left(\frac{2\pi}{\omega_2 - \omega_1} \right)$$

$$= 4 \left(\frac{2\pi}{0.45\pi - 0.3\pi} \right) = 53.33$$

So, $N = 53.33$ ~~53.33~~ let select the next odd integer after 53.33

$$\text{i.e. } N = 55$$

length of filter, $M = 55$

The ideal response is.

$$H_d(\omega) = \begin{cases} e^{j\omega \left(\frac{M-1}{2} \right)} & \text{for } -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

To obtain sample response is.

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(\omega) \cdot e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega \left(\frac{M-1}{2} \right)} \cdot e^{j\omega n} d\omega \end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega \left(n - \frac{M-1}{2} \right)} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega \left(n - \frac{M-1}{2} \right)}}{j \left(n - \frac{M-1}{2} \right)} \right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega_c \left(n - \frac{M-1}{2} \right)} - e^{-j\omega_c \left(n - \frac{M-1}{2} \right)}}{j \left(n - \frac{M-1}{2} \right)} \right]$$

$$= \frac{1}{\pi \left(n - \frac{M-1}{2} \right)} \left[\frac{e^{j\omega_c \left(n - \frac{M-1}{2} \right)} \cdot e^{-j\omega_c \left(n - \frac{M-1}{2} \right)}}{2j} \right]$$

$$h_d(n) = \frac{\sin \left[\omega_c \left(n - \frac{M-1}{2} \right) \right]}{\pi \left(n - \frac{M-1}{2} \right)} \quad n \neq \frac{M-1}{2}$$

$$\text{When } n = \frac{M-1}{2}$$

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$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{1}{2\pi} \left[\omega \right]_{-\omega_c}^{\omega_c}$$

$$= \frac{\omega_c}{\pi}$$

So,

$$h_d(n) = \begin{cases} \frac{\sin\left[\omega_c \left(n - \frac{M-1}{2}\right)\right]}{\pi \left(n - \frac{M-1}{2}\right)} & n \neq \frac{M-1}{2} \\ \frac{\omega_c}{\pi} & \text{for } n = \frac{M-1}{2} \end{cases}$$

Again,

eqn of hamming window is. (iv)

$$w(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{M-1}\right) \quad \text{(v)}$$

$$0 \leq n \leq M-1$$

Now, required solution is,

$$h(n) = h_d(n) \cdot w(n)$$

$$h(n) = \begin{cases} \frac{\sin\left[\omega_c \left(n - \frac{M-1}{2}\right)\right]}{\pi \left(n - \frac{M-1}{2}\right)} \cdot \left[0.54 - 0.46 \cos\left(\frac{2\pi n}{M-1}\right)\right] \\ \frac{\omega_c}{\pi} \cdot \left[0.54 - 0.46 \cos\left(\frac{2\pi n}{M-1}\right)\right] \end{cases}$$

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$$\text{for } n \neq \frac{M-1}{2}$$

$$\frac{\omega_c}{\pi} \left[0.54 - 0.46 \cos\left(\frac{2\pi n}{M-1}\right)\right] \quad \text{for } n = \frac{M-1}{2}$$

$$\text{here, } n = \frac{M-1}{2} = \frac{55-1}{2} = 27$$

$$\omega_1 = \omega_c = 0.3 \text{ rad/sec. So,}$$

$$h(n) = \begin{cases} \frac{\sin[0.3\pi(n-27)]}{\pi(n-27)} & \text{for } n \neq 27 \\ \frac{0.3\pi}{\pi} \left[0.54 - 0.46 \cos\left(\frac{\pi n}{27}\right)\right] & \text{for } n = 27 \end{cases}$$

$$0 \leq n \leq 54$$

#

Q.N. 2.

Here, given,

ideal response is,

$$h_d(\omega) = h_d(e^{j\omega}) = \begin{cases} 1 & \text{for } -\pi/6 \leq \omega \leq \pi/6 \\ 0 & \text{; otherwise.} \end{cases}$$

The filter length $(M) = 9$

$$\tau = \frac{M-1}{2} = 4$$

To obtained sample response.

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega}^{\omega} h_d(\omega) \cdot e^{j\omega(n-\tau)} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/6}^{\pi/6} 1 \cdot e^{j\omega(n-4)} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-4)}}{jn} \right]_{-\pi/6}^{\pi/6}$$

$$e^{j\omega(n-\tau)}$$

$$= \frac{1}{2j} \left[\frac{e^{j\pi/6(n-4)}}{1} - \frac{e^{-j\pi/6(n-4)}}{1} \right]$$

$$= \frac{1}{j} \left[\frac{\sin(\pi/6(n-4))}{\pi/6} \right]$$

$$h_d(n) = \frac{\omega_c}{\pi} = \frac{\pi/6}{\pi} = \frac{1}{6}$$

$$0-8$$

$$w(n) = 0.5 - 0.5 \cos\left(\frac{2\pi n}{M-1}\right)$$

$$= 0.5 - 0.5 \cos\left(\frac{\pi n}{4}\right)$$

$$0-8$$

$$h(n) = h_d(n) \cdot w(n)$$

FIR filter

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→ Impulse response settles to zero in finite time

→ It is always stable

→ H is generally linear phase.

→ H is non-recursive

→ mathematically,

$$y[n] = \sum_{k=0}^N b_k x[n-k]$$

uses only current and past inputs

IIR filter

→ Impulse response continues indefinitely.

→ H is unstable.

→ H is generally non-linear phase.

→ It is recursive.

→ mathematically,

$$y[n] = \sum_{k=0}^N b_k x[n-k] - \sum_{j=1}^M a_j y[n-j]$$

→ uses past inputs and past outputs.

kaiser window:

kaiser window is,

$$w(n) = \frac{I_0 \left\{ \beta \left(1 - \left| \frac{n-\alpha}{\alpha} \right|^2 \right) \right\}^{1/2}}{I_0(\beta)} \quad \text{for } 0 \leq n \leq M$$

When, $\alpha = \frac{M}{2}$

$$\beta = \begin{cases} 0.1102(A-8.7) & \text{for } A \geq 50 \\ 0.5842(A-2)^{0.4} + 0.07886(A-2) & \text{for } 21 \leq A \leq 50 \\ 0 & \text{for } A < 21 \end{cases}$$

$$M = \frac{A \cdot 8}{2.285 \Delta \omega}$$

$\Delta \omega \rightarrow$ transition width

$$\Delta \omega = \omega_s - \omega_p$$

finally,

$$h(n) = h_d(n) \cdot w(n)$$

$$\rightarrow A_p = 20 \log_{10} \left(\frac{1 + S_p}{1 - S_p} \right)$$

$$\rightarrow S = \frac{10^{0.05 A_p - k}}{10^{0.05 A_p - k}}$$

$$\rightarrow A_s = -20 \log_{10}(S)$$

$$\rightarrow A = -20 \log_{10}(S)$$

Here, given, $1.0.01 \leq |H(e^{j\omega})| \leq 1.0.01$

$$\text{for } 0 \leq |\omega| < 0.19\pi$$

$$|H(e^{j\omega})| \leq 0.01, \text{ for } 0.2\pi \leq |\omega| \leq \pi$$

Where

$$\begin{aligned} S_1 &= 0.01 & S_2 &= 0.01 \\ \omega_p &= 0.19\pi & \omega_s &= 0.2\pi \end{aligned}$$

$$A\omega = \omega_p = 0.19\pi = 0.19\pi = 0.19\pi$$

$$S = \text{minimum of } S_1 \text{ and } S_2$$

$$S = 0.01$$

$$\begin{aligned} A &= -20 \log_{10}(S) \\ &= -20 \log_{10}(0.01) = 40 \end{aligned}$$

find cut off frequency,
 $\omega_c = \omega_p + \omega_s = 0.2\pi$

Here $A = 40$. It lies in $21 \leq A \leq 50$

$$\begin{aligned} \beta &= 0.5842(A-1)^{0.7} + 0.0794(A-1) \\ &= 2.395 \end{aligned}$$

Again, $M = \frac{A-1}{2.395} = \frac{40-1}{2.395} = 16.72$

Again, $\alpha = \frac{M}{2} = \frac{222.88}{2} = 111.5$

$\beta = 3.395$

So. The Kaiser window is.

$$w(n) = \frac{I_0 \left[\beta \left(1 - \frac{\alpha - 1}{2} \right)^2 \right]^{1/2}}{I_0(\beta)} \quad \text{for } 0 \leq n \leq 223.$$

ideal desired frequency response.

$$h_d(\omega) = h_d(e^{j\omega}) = \begin{cases} e^{-j\omega \left(\frac{M-1}{2} \right)} & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{o.t.h.} \end{cases}$$

to obtained sample response,

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} h_d(e^{j\omega}) \cdot e^{j\omega n} d\omega.$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega \left(\frac{M-1}{2} \right)} \cdot e^{j\omega n} d\omega$$

$$= \frac{\sin \left[\omega_c \left(n - \frac{M-1}{2} \right) \right]}{\pi \left(n - \frac{M-1}{2} \right)} \quad n \neq \frac{M-1}{2}$$

$$h_d(n) = \begin{cases} \frac{\sin \left[\omega_c \left(n - \frac{M-1}{2} \right) \right]}{\pi \left(n - \frac{M-1}{2} \right)} & \text{for } n \neq \frac{M-1}{2} \\ \omega_c & \text{for } n = \frac{M-1}{2} \end{cases}$$

filter length $(M+1) = 223+1 = 224$

$$h_d(n) = \frac{\sin \left(0.21\pi \left(n - \frac{223}{2} \right) \right)}{\pi \left(n - \frac{223}{2} \right)}$$

$$= \frac{\sin \left[0.21\pi (n - 111.5) \right]}{\pi (n - 111.5)} \quad \text{for } 0 \leq n \leq 223$$

Now, The required solution.

$$h(n) = h_d(n) \cdot \omega(n)$$

$$= \left\{ \frac{\sin(0.2\pi(n-11.5))}{\pi(n-11.5)} \right\} \cdot \left\{ 3.345 \left(1 - \frac{n-11.5}{11.5} \right)^{2/2} \right\}$$

$$n = 60; \text{ for } 0 \leq n \leq 223; \text{ otherwise.}$$

$$\textcircled{2} \quad h_d(e^{j\omega}) = \begin{cases} 1 & : -\pi/6 \leq \omega \leq \pi/6 \\ 0 & : \text{otherwise.} \end{cases}$$

$$M = 9 \quad \left(\tau = \frac{M-1}{2} = 4 \right)$$

to find, impulse response,

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega}^{\omega} h_d(e^{j\omega}) \cdot e^{j\omega(n-\tau)} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/6}^{\pi/6} e^{j\omega(n-\tau)} d\omega$$

$$(\cos \theta + j \sin \theta)$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)} \right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\pi/6(n-\tau)} - e^{-j\pi/6(n-\tau)}}{j(n-\tau)} \right]$$

$$= \frac{1}{\pi(n-\tau)} \left[\frac{e^{j\pi/6(n-\tau)} - e^{-j\pi/6(n-\tau)}}{2j} \right]$$

$$= \frac{\sin \pi/6(n-\tau)}{\pi(n-\tau)} \quad \left[\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \right]$$

$$h_d(n) = \begin{cases} \frac{\sin \pi/6(n-\tau)}{\pi(n-\tau)} & : n \neq \tau = \frac{M-1}{2} \\ 1 & : n = \tau = \frac{M-1}{2} \end{cases}$$

$$h_d(n) = \frac{\sin \pi/6(n - \frac{M-1}{2})}{\pi(n - \frac{M-1}{2})} \quad \text{for } n \neq \frac{M-1}{2}$$

$$n = \frac{M-1}{2}$$

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$$h_d(n) = \frac{\sin \frac{\pi}{6} (n-2)}{\pi (n-2) \times 6}$$

$$= \frac{1}{6} \quad n=2$$

So,

$$h_d(n) = \begin{cases} \frac{\sin \frac{\pi}{6} (n - \frac{M-1}{2})}{\pi (n - \frac{M-1}{2})} & \text{for } n \neq 2 \\ \frac{1}{6} & \text{for } n = 2 \end{cases}$$

0
1
2 3, 5, 6, 7, 8

$$h_d(0) \dots h_d(8)$$

Again, for hanning window

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$$w(n) = 0.5 - 0.5 \cos \left(\frac{2\pi n}{M-1} \right)$$

$$= 0.5 - 0.5 \cos \left(\frac{\pi n}{4} \right)$$

$$w(0) \dots w(8)$$

Now, The require solution.

$$h(n) = h_d(n) \cdot w(n)$$

$$h(0) = \underline{\hspace{2cm}}$$

3(a) Here.

$$h(z) = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + 3.5z^{-1} + 1.5z^{-2}}$$

~~numerator~~

$$= \frac{1 + \frac{3}{z} + \frac{2}{z^2}}{1 + \frac{3.5}{z} + \frac{1.5}{z^2}}$$

$$= \frac{z^2 + 3z + 2}{z^2 + 3.5z + 1.5}$$

for Numerator: $z^2 + 3z + 2 = (z+1)(z+2)$

denominator: $z^2 + 3.5z + 1.5 = (z+3)(z+0.5)$

So,

$$h(z) = \frac{(z+1)(z+2)}{(z+3)(z+0.5)} \quad \text{--- } \textcircled{i}$$

By Partial fraction,

$$\frac{(z+1)(z+2)}{(z+3)(z+0.5)} = \frac{A}{(z+3)} + \frac{B}{(z+0.5)}$$

$$(z+1)(z+2) = A(z+0.5) + B(z+3)$$

At $z = -3$,

$$-2 \times (-1) = -2.5A$$

$$2 = -2.5A$$

$$\therefore A = -0.8$$

At $z = -0.5$,

$$0.5 \times 1.5 = 2.5B$$

$$\therefore B = 0.3$$

eqⁿ $\textcircled{i}$ becomes.

$$H(z) = 1 - \frac{0.8}{(z+3)} + \frac{0.3}{(z+0.5)}$$

$$\frac{h(z)}{\textcircled{2}} = 1 - \frac{0.8z^{-1}}{(1+3z^{-1})} + \frac{0.3z^{-1}}{(1+0.5z^{-1})}$$

Taking inverse z-transform, we get.

$$h(n) = \delta(n) - 0.8(-3)^{n-1} u[n-1] + (0.3)^{n-1} u[n-1]$$

for ROC:

$|z| > 3$ for causal

other possible ROC,

1. $|z| > 3$ for causal
2. $|z| < 0.5$ for anti-causal
3. $0.5 < |z| < 3$ for two-sided.

~~2a)~~ here.

$$x[n] = \{ 5(2^n) - 9(3^n) \} u[n]$$
$$= 5 \cdot 2^n u[n] - 9 \cdot 3^n u[n] \quad \text{--- (i)}$$

By scaling property of z-transform,
we know,

$$a^n u[n] \xrightarrow{z} \frac{1}{1 - az^{-1}}$$

so, eqn (i),

$$X(z) = 5 \cdot \frac{1}{1 - 2z^{-1}} - 9 \cdot \frac{1}{1 - 3z^{-1}}$$
$$= \frac{5}{1 - 2z^{-1}} - \frac{9}{1 - 3z^{-1}}$$

for ROC:

$|z| > 2$

$|z| > 3$

2b) here. Given.

$$X(z) = \frac{1}{1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$= \frac{1}{1 + \frac{1.5}{z} + 0.5 \frac{1}{z^2}}$$

$$= \frac{z^2}{z^2 + 1.5z + 0.5}$$

$$= \frac{z^2}{(z+1)(z+0.5)} \quad \text{--- (ii)}$$

By partial fraction,

$$X(z) = \frac{A}{(z+1)} + \frac{B}{(z+0.5)} \quad \text{--- (i)}$$

$$\frac{2}{z} = A(z+0.5) + B(z+1)$$

$$\text{At } z = -1,$$

$$1 = -0.5A$$

$$\therefore A = -2$$

$$\text{At } z = -0.5,$$

$$0.25 = 0.5B$$

$$\therefore B = 0.5$$

eqⁿ (i) becomes,

$$X(z) = \frac{-2}{(z+1)} + \frac{0.5}{(z+0.5)}$$

$$= -2 \cdot \left(\frac{z^{-1}}{1 + z^{-1}} \right) + 0.5 \left(\frac{z^{-1}}{1 + 0.5z^{-1}} \right)$$

Taking inverse z-transform,

$$x(n) = -2(-1)^n u(n-1) + 0.5(-0.5)^{n-1} u(n-1)$$

for. for:

$|z| > 1$ for causal
 $|z| < 1/2$ for anti causal.

$$2b] \quad x(n) = \left[\left(\frac{1}{2} \right)^n + \left(\frac{3}{4} \right)^n \right] \cdot 4(n-5)$$

Taking z-transform on both side,

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left[\left(\frac{1}{2} \right)^n + \left(\frac{3}{4} \right)^n \right] 4(n-5) \right\} z^{-n}$$

We know,

$$u[n-5] = 1 \quad ; \quad n-5 \geq 0 \text{ i.e. } n \geq 5$$

$$0 \quad ; \quad \text{otherwise.}$$

eqⁿ (i) becomes,

$$X(z) = \sum_{n=5}^{\infty} \left\{ \left(\frac{1}{2} \right)^n + \left(\frac{3}{4} \right)^n \right\} \cdot 1 \cdot z^{-n}$$

$$= \sum_{n=5}^{\infty} \left(\frac{1}{2} z^{-1} \right)^n + \sum_{n=5}^{\infty} \left(\frac{3}{4} z^{-1} \right)^n$$

$$= \left(\frac{1}{2} z^{-1} \right)^5 + \left(\frac{1}{2} z^{-1} \right)^6 + \left(\frac{1}{2} z^{-1} \right)^7 + \dots$$

$$+ \left(\frac{3}{4} z^{-1} \right)^5 + \left(\frac{3}{4} z^{-1} \right)^6$$

$$+ \left(\frac{3}{4} z^{-1} \right)^7 + \dots$$

We know, $\sum_{k=0}^{\infty} r^k$

So..

$$X(z) = \frac{\left(\frac{1}{2}z^{-1}\right)^5}{1 - \frac{1}{2}z^{-1}} + \frac{\left(\frac{3}{4}z^{-1}\right)^5}{1 - \frac{3}{4}z^{-1}}$$

$$= \frac{\left(\frac{1}{2}\right)^5 z^{-5} + \left(\frac{3}{4}\right)^5 z^{-5}}{\left(1 - \frac{1}{2}z^{-1}\right) \left(1 - \frac{3}{4}z^{-1}\right)}$$

$$= \frac{\left(\frac{1}{2}\right)^5 z^{-4}}{\left(z + \frac{1}{2}\right)} + \frac{\left(\frac{3}{4}\right)^5 z^{-4}}{\left(z + \frac{3}{4}\right)}$$

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Remez Exchange Algorithm.

choose the value of $(L+1)$ external frequency

calculate the optimal ripples on external set.

Interpolate through $(L+2)$ point to obtain amplitude response, $A(\omega)$

calculate error, $E(\omega)$ and find the local maxima, where $|E(\omega)| \geq \delta$

more than $(L+2)$ external points

yes

Return $(L+2)$ largest extreme

No

extreme point change

Best approximation

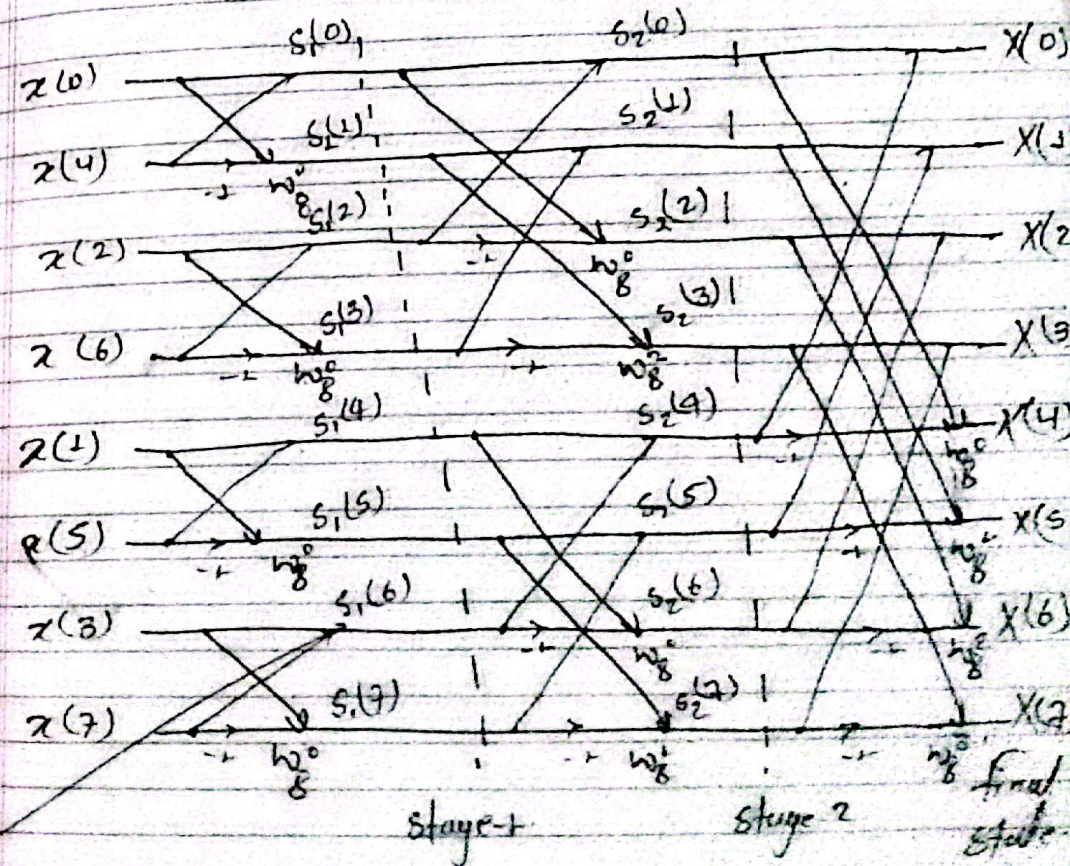


fig. 8. point

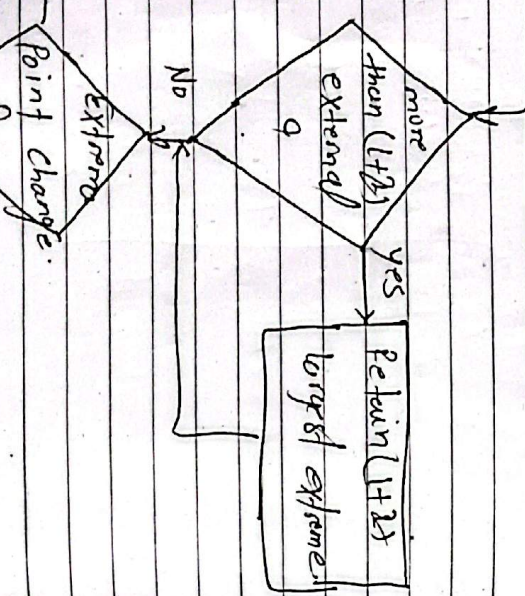
Perturb Exchange Algorithm.

Choose the value of $L^{(1)}$ external frequency

calculate the optimal signals on external set.

Interpolate through $(L^{(1)}, P_{int})$ to obtain amplitude response, $A(\omega)$

calculate error, $E(\omega)$ and find the local maxima, where $|E(\omega)| \geq \delta$



Best approximation

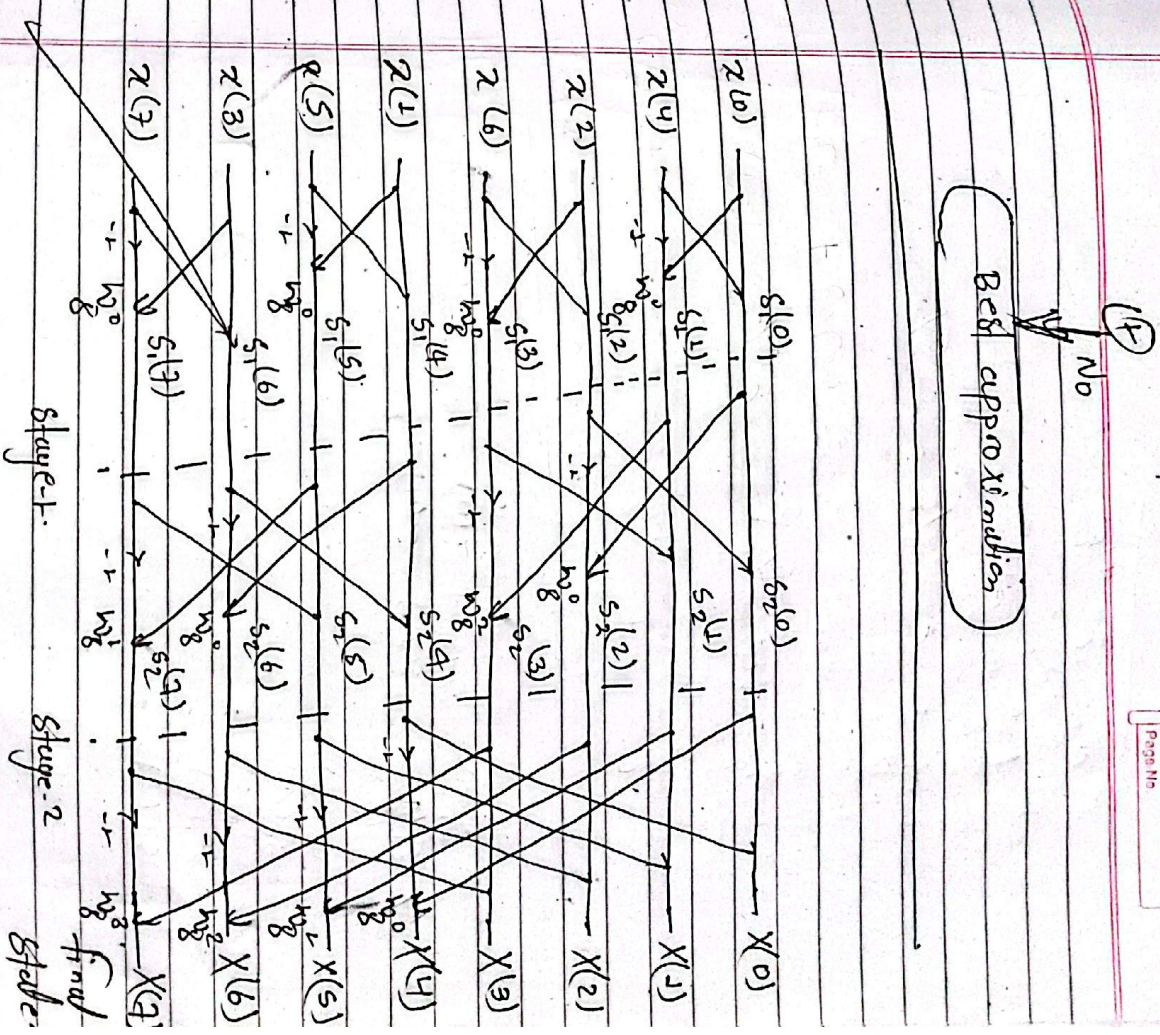


fig: 8-point.