

New Syllabus

Insights on
**Applied
PHYSICS**

Third Edition

By:

SHANKAR PARAJULI

(DHOD, Dept. of Applied Sciences, KEC, Kalimati)

System Inception

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CHAPTER 1

MECHANICAL OSCILLATION

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1.1 Simple Harmonic Motion

Any to-and-fro motion in which the acceleration of an object is directly proportional to its displacement from the mean position and directed towards the mean position is called *simple harmonic motion*. There are many examples of simple harmonic motion in nature such as motion of simple pendulum, compound pendulum, torsion pendulum, mass-spring system, etc. Mathematical analysis of simple harmonic motion needs following terminology:

- **Displacement:**

Distance of a particle from the mean position is called displacement of a particle. It can be expressed as sine or cosine function; $x = A \sin(\omega t + \phi)$, where A is the maximum displacement or amplitude, ω is angular velocity and ϕ is the phase difference of wave.

- **Velocity (v):**

Rate of change of displacement is called velocity, which is given by

$$v = \frac{dx}{dt} = A\omega \cos(\omega t + \phi)$$

$$\text{or, } v = \omega \sqrt{A^2 - x^2}$$

Velocity is maximum at mean position i.e., at $x = 0$ and the maximum velocity is

$$v_{\max} = A\omega$$

Velocity is minimum at extreme position where $x = A$ and the minimum velocity is

$$v_{\min} = 0$$

- **Acceleration (a):**

Rate of change of velocity is called acceleration, which is given by

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A$$

Acceleration is maximum at extreme position i.e. at $x = A$ and the maximum acceleration is

$$a_{\max} = \omega^2 A$$

Acceleration is minimum at mean position i.e., at $x = 0$ and the minimum acceleration is

$$a_{\min} = 0$$

Simple harmonic motion may be *linear* or *angular*. The differences between linear harmonic motion and angular harmonic motion are as follows:

Linear harmonic motion	Angular harmonic motion
1. Here, acceleration produced on particle at any point is directly proportional to the linear displacement.	1. Here, acceleration produced on particle at any point is directly proportional to the angular displacement.
2. Restoring force is provided by weight, force constant. Example: mass-spring system	2. Restoring torque is provided by torsional constant. Example: torsional pendulum

1.1.1 Kinetic Energy and Potential Energy in Simple Harmonic Motion

The displacement of a particle executing SHM is given by

$$x = A \sin(\omega t + \phi) \quad \dots\dots\dots(i)$$

Then, velocity of the particle can be calculated as

$$v = \frac{dx}{dt} = A\omega \cos(\omega t + \phi)$$

$$\text{Kinetic energy (K)} = \frac{1}{2}mv^2$$

$$= \frac{1}{2}mA^2\omega^2 \cos^2(\omega t + \phi)$$

$$\text{Potential energy (U)} = \frac{1}{2}kx^2$$

$$= \frac{1}{2}(m\omega^2)[A\sin(\omega t + \phi)]^2$$

$$= \frac{1}{2}m\omega^2A^2 \sin^2(\omega t + \phi)$$

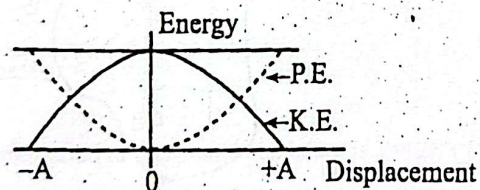
$$\text{Total energy (E)} = K + U$$

$$= \frac{1}{2}mA^2\omega^2 \cos^2(\omega t + \phi) + \frac{1}{2}m\omega^2A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}m\omega^2A^2 = \frac{1}{2}kA^2$$

which is the expression for total energy of mass-spring system executing SHM. Hence, the total energy in SHM is a constant quantity.

The variation of K and U is shown in graph below.



When $x = \pm A$, $k = 0$ and total energy equals maximum potential energy.

$$E = U_{\max}$$

$$= \frac{1}{2}kA^2$$

Similarly, when $x = 0$, $U = 0$ and total energy equals maximum kinetic energy.

$$E = K_{\max}$$

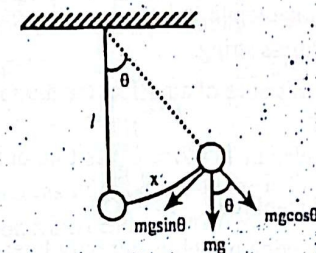
$$= \frac{1}{2}kA^2$$

1.2 Simple Pendulum

A heavy point object suspended from a massless string oscillating freely through a fixed point is called *simple pendulum*. However, these requirements cannot be achieved in practice. Hence, a simple pendulum is called an *ideal pendulum*.

1.2.1 Time Period of a Simple Pendulum

Consider of bob of mass m is suspended from a string of length l as shown in figure.



Let the bob is displaced by small angle θ and released. Then, it starts to oscillate in simple harmonic motion.

Force acting on bob is given by

$$m \frac{d^2x}{dt^2} = -mg \sin \theta$$

$$\text{or, } \frac{d^2(\theta)}{dt^2} + g \sin \theta = 0$$

For small angle $\sin \theta \approx \theta$, so

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \theta = 0$$

$$\text{or, } \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

which is the differential equation of simple harmonic motion with angular velocity

$$\omega = \sqrt{\frac{g}{l}}$$

$$\text{or, } \omega = 2\pi f = \sqrt{\frac{g}{l}}$$

$$\text{or, } \frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

$$\therefore T = 2\pi\sqrt{\frac{l}{g}}$$

which is the required equation for time period of simple pendulum.

Drawbacks of simple pendulum:

- It is impossible to have both the point mass of the bob and weightless string.
- The resistance of air affect the motion of bob.
- The relation $T = 2\pi\sqrt{\frac{l}{g}}$ is true only for oscillation of very small amplitude.
- The motion of bob is not only linear but it has also rotatory motion which affects the time period of simple pendulum.

1.2.2 Time Period of Mass-Spring System

Consider a block of mass m is attached at the end of the spring of spring constant k . when the spring is extended by small distance x and released then the system starts to oscillate about its mean position.

We know the restoring force on the spring $F = -kx$ (i)

$$\text{or, } m\frac{d^2x}{dt^2} = -kx$$

$$\text{or, } \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\text{or, } \frac{d^2x}{dt^2} + \omega^2x = 0 \dots\dots(ii)$$

which is the differential equation of simple harmonic motion

$$\text{and } \omega^2 = \frac{k}{m}$$

$$\text{or, } \omega = \sqrt{\frac{k}{m}}$$

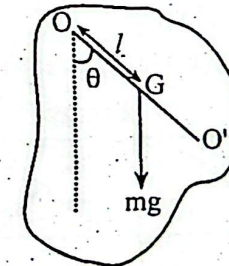
$$\text{or, } \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\therefore T = 2\pi\sqrt{\frac{m}{k}} \dots\dots(iii)$$

This is the required expression for time period of mass - spring system.

1.3 Physical Pendulum

Any rigid body which can oscillate freely through a fixed point is called *physical pendulum* or *compound pendulum*. Bar pendulum is an example of physical pendulum.



Consider a rigid object of mass m be suspended at point O. The centre of gravity of the object is G at distance l from O since l is the distance from the pivot to centre of mass.

When the object is displaced by an angle θ , there are two forces acting on the object:

- i. Force due to gravity (mg)
- ii. The restoring force (which tries to bring the object towards the mean position)

From rotational dynamics, torque produced is

$$\tau = I \alpha$$

Here, Torque (τ) = Force $\times$ perpendicular distance between the forces

$$\text{or, } I \alpha = -mg \times l \sin \theta$$

$$\text{or, } -mg / \sin \theta = I \alpha$$

where I is moment of inertia of object about the given axis and α is angular acceleration.

$$-mg l \sin \theta = I \frac{d^2 \theta}{dt^2}$$

$$\text{or, } \frac{d^2 \theta}{dt^2} + \frac{mg l}{I} \sin \theta = 0$$

For small θ , $\sin \theta \approx \theta$, so

$$\frac{d^2 \theta}{dt^2} + \frac{mg l}{I} \theta = 0$$

$$\text{or, } \frac{d^2 \theta}{dt^2} + \omega^2 \theta = 0$$

$$\text{where } \omega^2 = \frac{mg l}{I}$$

$$\text{Time period of compound pendulum is } T = 2\pi \sqrt{\frac{I}{mg l}}$$

This is the equation for simple harmonic motion indicating that physical pendulum executes simple harmonic motion.

Moment of inertia of pendulum about its axis passing through O and perpendicular to it is given by

$$I = mk^2 + ml^2$$

where k is the radius of gyration.

$$\text{or, } \frac{d^2 \theta}{dt^2} + \frac{mg l}{mk^2 + ml^2} \theta = 0$$

Comparing this equation with differential equation of SHM, time period will be

$$T = 2\pi \sqrt{\frac{k^2 + l^2}{lg}}$$

$$\text{or, } T = 2\pi \sqrt{\frac{\frac{k^2}{l} + l}{g}}$$

which is the required expression for time period of physical pendulum. The quantity $\frac{k^2}{l} + l$ is called *equivalent length of the simple pendulum*.

1.3.1 Minimum Time Period of Compound Pendulum

Time period of compound pendulum is given by

$$T = 2\pi \sqrt{\frac{\frac{k^2}{l_1} + l_1}{g}}$$

where k is radius of gyration, and l_1 is the distance of point of suspension from centre of gravity.

The time period will be minimum if $\frac{k^2}{l_1} + l_1$ is minimum.

So, differentiating $\frac{k^2}{l_1} + l_1$ with respect to l_1 , we have

$$\frac{d}{dl_1} \left[\frac{k^2}{l_1} + l_1 \right] = 0$$

$$\text{or, } \frac{-k^2}{l_1^2} + 1 = 0$$

$$\text{or, } \frac{k^2}{l_1^2} = 1$$

$$\text{or, } k^2 = l_1^2$$

$$\therefore k = l_1$$

1.3.2 Interchangeability of Point of Suspension and Oscillation

Let time period be T_1 for point A at distance l_1 from centre of gravity. Similarly, time period be T_2 for a point C at distance l_2 from centre of gravity as shown in figure.



$$T_1 = 2\pi \sqrt{\frac{\frac{k^2}{l_1} + l_1}{g}}$$

$$T_2 = 2\pi \sqrt{\frac{\frac{k^2}{l_2} + l_2}{g}}$$

Since $k^2 = l_1 l_2$,

$$\frac{k^2}{l_1} = l_2 \text{ and } \frac{k^2}{l_2} = l_1.$$

This makes $T_1 = T_2$.

This shows that point of suspension and oscillation are interchangeable.

1.3.3 Four Collinear Points in Compound Pendulum

Time period of a compound pendulum is given by

$$T = 2\pi \sqrt{\frac{\frac{k^2}{l} + l}{g}}$$

$$\text{or, } T^2 = 4\pi^2 \left(\frac{k^2 + l^2}{lg} \right)$$

$$\text{or, } T^2 l g = 4\pi^2 k^2 + 4\pi^2 l^2$$

$$\text{or, } 4\pi^2 l^2 - gT^2 l + 4\pi^2 k^2 = 0$$

which is quadratic in l . So, it has two values, say l_1 and l_2 .

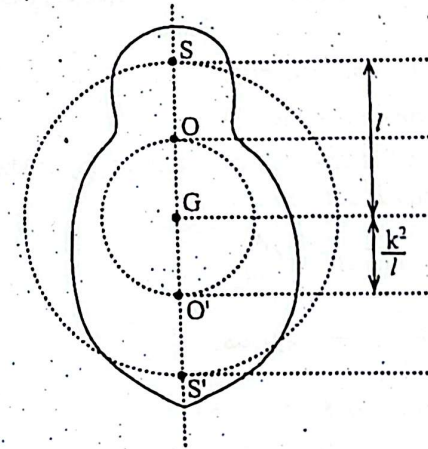
Using formula for sum of roots and product of roots, we can write

$$l_1 + l_2 = \frac{gT^2}{4\pi^2} \text{ and}$$

$$l_1 l_2 = \frac{4\pi^2 k^2}{4\pi^2} = k^2$$

If $l_1 = \frac{k^2}{l}$, then $l_2 = l$ or vice-versa.

In fact, there are two other points on either side of G about which time period will be same as about S and O' .



Draw two circles of radii $\frac{k^2}{l}$ and l having the centre of gravity G . The straight line through G will intersect the circles at points S , O' , O , and S' . Thus, four points are collinear with CG about which time period is same.

From the figure,

$$SG = GS' = l$$

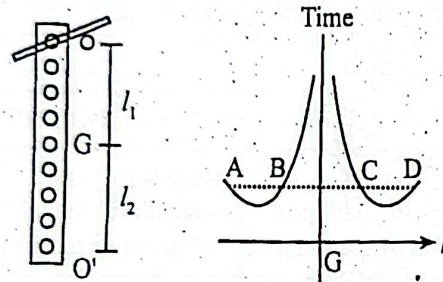
$$GO' = GO = \frac{k^2}{l}$$

$$\text{Hence, } OS = \frac{k^2}{l} + l = O'S'$$

where $\frac{k^2}{l} + l = L$ is equivalent length of simple pendulum.

1.3.4 Determination of Radius of Gyration and Acceleration Due to Gravity Using Bar Pendulum

A bar pendulum consists of a metal bar with large number of holes as shown in figure. Radius of gyration is defined as the perpendicular distance of the axis of rotation from a point at which the total mass of the body may be assumed to be concentrated.



Time period can be measured by fixing the knife edge in each corresponding holes. The distance of each hole from the centre of gravity is measured. If a graph between distance from centre of gravity and corresponding time period is plotted, it looks like as shown in figure.

Two points not equal from centre of gravity G in bar pendulum O and O' on both sides of it, one corresponding to centre of suspension and other as centre of oscillation. Let their distance be l_1 and l_2 from the centre of gravity as shown. Thus, the equivalent length of the pendulum is $L = L_1 + L_2$.

The time period of compound pendulum is given by

$$T = 2\pi \sqrt{\frac{k^2 + l_1^2}{l_1 g}}$$

$$= 2\pi \sqrt{\frac{k^2}{l_1} + l_1} \dots \dots (1)$$

Comparing equation (i) with $T = 2\pi \sqrt{\frac{L}{g}}$, we get

$$L = \frac{k^2}{l_1} + l_1$$

$$\text{or, } l_1 + l_2 = \frac{k^2}{l_1} + l_1$$

$$\text{or, } k^2 = l_1 l_2$$

$$\text{or, } k = \sqrt{l_1 l_2}$$

In this way, the value of radius of gyration k about the axis passing through the centre of gravity of a bar pendulum can be determined.

Determination of acceleration due to gravity from compound pendulum is more accurate due to following reasons:

- It is not an ideal concept i.e., it can be realized in actual practice.
- There is no lag like that in between bob and string in case of simple pendulum.
- Distance between two knife edges can be easily measured but it is difficult to measure the accurate length of simple pendulum.
- Due to larger mass and hence, larger moment of inertia, it continues to oscillate for a longer time.

1.4 Torsional Pendulum

A heavy body like a cylinder or a disc, fastened at its mid-point to a fairly long and thin wire suspended from a rigid support constitutes a *torsional pendulum*.

Torsion pendulum consists of a disc suspended at the end of a wire, as shown in figure other end of the wire is fixed at a rigid support. When the disc is rotated, the wire is twisted by an angle θ . Then restoring torque is created on it obeying Hook's law. The restoring torque is directly proportional to the angular displacement of wire.

$$\tau \propto \theta$$



$$\tau = -C\theta \dots\dots\dots (i)$$

where C is called couple per unit twist or torsional constant and is given by $C = \frac{\pi\eta r^4}{2l}$, r is radius and l is length of the wire.

Rotational form of Newton's 2nd law is

$$\tau = I \alpha$$

$$\text{or, } \tau = I \frac{d^2\theta}{dt^2}$$

From equation (i),

$$-C\theta = I \frac{d^2\theta}{dt^2}$$

$$\text{or, } \frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

$$\text{or, } \frac{d^2\theta}{dt^2} + \omega^2\theta = 0; \omega^2 = \frac{C}{I}$$

This is the equation of simple angular harmonic motion. Hence, the motion of torsional pendulum is angular harmonic.

$$\text{Angular frequency } (\omega) = \sqrt{\frac{C}{I}}$$

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{C}}$$

which is the required expression for time period of torsional pendulum.

1.4.1 Applications of Torsion Pendulum

I. Determination of Modulus of Rigidity of Wire

To determine the modulus of rigidity of given wire, first of all the wire is suspended from a rigid support with a circular disc. Time period of torsion oscillation is taken for particular length l. Let the time period be T₁. Time period of torsion oscillation is again taken by placing a circular ring of known moment of inertia over the disc let the time period be T₂.

$$\text{We know time period with disc is } T_1 = 2\pi \sqrt{\frac{I}{C}} \dots\dots (1)$$

where I is moment of inertia of disc and C is the torsion constant of the wire.

Again, time period with disc and ring is

$$T_2 = 2\pi \sqrt{\frac{I + I_0}{C}} \dots\dots\dots (2)$$

where I₀ is the moment of inertia of the ring.

Squaring and subtracting (1) from (2),

$$T_2^2 - T_1^2 = \frac{4\pi^2 I_0}{C}$$

$$T_2^2 - T_1^2 = \frac{4\pi^2 I_0}{\frac{\pi\eta r^4}{2l}}$$

$$\text{where } C = \frac{\pi\eta r^4}{2l}$$

$$\eta = \frac{8\pi I_0 l}{(T_2^2 - T_1^2) r^4} \dots\dots\dots (3)$$

By measuring the time period, length of wire and moment of inertia of ring, we can determine the modulus of rigidity of wire.

1.5 Similarities and Dissimilarities between the Oscillations of Bar Pendulum & Torsional Pendulum

Similarities:

- i. In both pendulum, the motions are simple harmonic.
- ii. Both pendulum can be used to measure the moment of inertia of a body.
- iii. Both pendulum oscillate about the centre of gravity.

Dissimilarities:

- Bar pendulum is used to measure acceleration due to gravity while torsional pendulum is used to measure modulus of rigidity.
- Amplitude of oscillation does not affect the motion of torsional pendulum while it affects the compound pendulum.
- There are two points having same time period in compound pendulum whereas there are not such two points in torsional pendulum.

Bar pendulum	Torsion pendulum
1. In bar pendulum, amplitude of oscillation is assumed to be small.	1. In torsion pendulum, amplitude of oscillation is not assumed to be small.
2. This pendulum can be used to measure moment of inertia, acceleration due to gravity, etc.	2. This pendulum can be used to measure the rigidity of wire.

1.6 Types of Oscillation

1.6.1 Free Oscillation

The motion in which particle moves without any restriction of the force i.e., motion of the particle is not retarded by external force such as friction etc. is called *free oscillation*. Examples: Motion of electrons around nucleus, motion of simple pendulum in vacuum, etc.

1.6.2 Damped Harmonic Mechanical Oscillation

Most of the oscillating bodies in daily life undergo reduction of amplitude due to the friction and various other resistive force; such oscillation is known as *damped harmonic oscillation*. The effect of damping decreases the energy and frequency of an oscillating object and ultimately stops the oscillation. The resistive force acting on particle executing

S.H.M. is proportional to the velocity of the particle which can be expressed as

$$F = -b \frac{dx}{dt} \dots\dots\dots (1)$$

where b is damping constant.

Therefore, differential equation in this case will be

$$F = -kx - b \frac{dx}{dt}$$

$$\text{or, } m \frac{d^2x}{dt^2} = -kx - b \frac{dx}{dt}$$

$$\text{or, } m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

$$\text{or, } \frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0$$

$$\text{or, } \frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0 \dots\dots\dots (2)$$

where $2\gamma = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$ are constant.

which is the second order differential equation of damped harmonic mechanical oscillation.

The solution of equation (2) can be obtained from auxiliary equation and is given as

$$(D^2 + 2\gamma D + \omega_0^2) x = 0$$

$$\text{or, } (D^2 + 2\gamma D + \omega_0^2) = 0$$

Solution of this quadratic equation is

$$D = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2}$$

$$= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

There are three possible scenarios:

Case I: roots, real and different (overdamping)

$$\gamma^2 > \omega_0^2$$

$$\text{or } \frac{b^2}{4m^2} > \frac{k}{m}$$

When $\frac{b^2}{4m^2} > \frac{k}{m}$, the damped frequency is imaginary. In this case, there is no oscillation and the system is said to be *overdamped*. And the displacement of oscillation is

$$x = A_1 e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + A_2 e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t}$$

where A_1 and A_2 are constant to be determine.

Case II: roots, real and equal (critical damping)

$$\gamma^2 = \omega_0^2 \text{ or } \frac{b^2}{4m^2} = \frac{k}{m}$$

When $\frac{k}{m} = \frac{b^2}{4m^2}$, then frequency vanishes. In this case, there is no oscillation and the system is said to be *critically damped*. Critical damping is highly desirable in mechanical suspension system of motor vehicles. Amplitude of critically damped oscillation is given by;

$$x = (A_1 + A_2 t) e^{-\gamma t}$$

where A_1 and A_2 are constant to be determined.

Case III: roots, imaginary (underdamped)

$$\omega_0^2 > \gamma^2$$

$$\text{or } \frac{k}{m} > \frac{b^2}{4m^2}$$

When $\frac{k}{m} > \frac{b^2}{4m^2}$, the damped frequency is real and the oscillation is said to be *underdamped* with damped angular

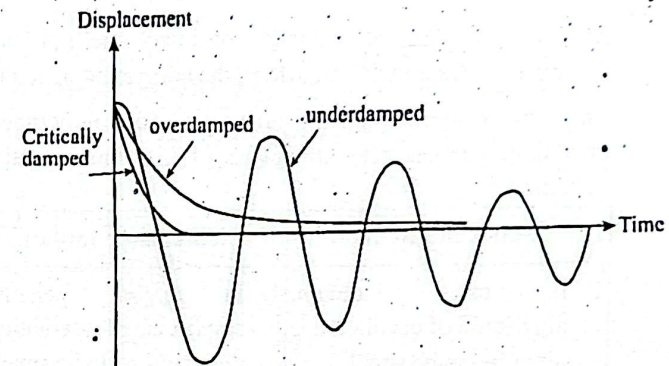
$$\text{frequency, } \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Hence, the damped frequency is

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

The amplitude of underdamped oscillation is given by

$$x = A e^{-\gamma t} \sin(\omega t + \phi_0) \dots\dots\dots (3)$$



Quality factor (Q)

The rate of energy loss of weakly damped oscillator is best characterized by a single parameter Q called the *Quality factor* of a damped oscillator. It is defined to be 2π times the energy stored in the oscillator divided by the energy lost in a single period of oscillation. Less is the damping, higher is the quality factor Q . Mathematically, it is defined as

$$Q = 2\pi \times \frac{\text{Energy stored}}{\text{Energy lost per cycle}}$$

If ω_1 and ω_2 be the frequencies at which the amplitude becomes $\frac{A_0}{\sqrt{2}}$, then the quantity $\omega_1 - \omega_2$ is called width of resonance peak. Thus, quality is related with $\omega_1 - \omega_2$ by

$$\frac{\Delta\omega}{\omega} = \frac{1}{Q} \dots\dots\dots (1)$$

The energy per cycle of damped harmonic motion is

$$E_{avg} = \frac{1}{2} m \omega^2 x_m^2 e^{\frac{-bt}{m}} \dots\dots\dots(2)$$

where b = damping constant.

As $\frac{b}{m} = 2\gamma = \frac{1}{\tau}$, 2γ is logarithmic decrement and τ is relaxation time. Then, above equation becomes

$$E_{avg} = \left[\frac{1}{2} m \omega^2 x_m^2 \right] e^{-2\gamma t} \\ = E_0 e^{-2\gamma t} \dots\dots\dots(3)$$

Average power dissipation is

$$P = -\frac{dE_{avg}}{dt} \\ = -\frac{d}{dt} (E_0 e^{-2\gamma t}) \\ = 2\gamma E_0 e^{-2\gamma t}$$

$$\text{or, } P = 2\gamma E_{avg} \dots\dots\dots(4)$$

$$\therefore \frac{E_{avg}}{P} = \frac{1}{2\gamma} \dots\dots\dots(5)$$

$$Q = 2\pi \frac{\text{Energy stored}}{\text{Energy loss per period}}$$

$$\text{or, } Q = 2\pi \frac{E_{avg}}{P T}$$

$$\text{or, } Q = \omega \frac{E_{avg}}{P}$$

$$\text{or, } Q = \frac{\omega}{2\gamma}$$

$$\text{or, } Q = \omega \tau$$

$$\therefore Q = \frac{m\omega}{b} \dots\dots\dots(6) \quad \left[\because \tau = \frac{m}{b} \right]$$

This is the expression for quality factor of damped harmonic oscillator. From equation (6), it is clear that quality factor is inversely proportional to damping constant b .

1.6.3 Forced Oscillation

In actual practice, oscillation left to itself eventually die out after certain times. But by applying periodic external force, we can maintain a constant amplitude. The phenomenon of setting body into continuous vibrations with the help of strong periodic force having frequency different from the natural frequency of the body is called *forced vibration*.

Let the applied force be represented as

$$F_{ext} = F_0 \sin pt \dots\dots\dots(1)$$

where F_0 is maximum force amplitude and p is frequency of applied force.

Newton's second law of motion in this case is

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \sin pt \dots\dots\dots(2)$$

$$\text{or, } \frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = F_0 \sin pt$$

$$\text{or, } \frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = F_0 \sin pt \dots\dots\dots(2)$$

where $2\gamma = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$ are constant.

After applying the sinusoidal external force, the oscillator oscillates with forcing frequency of $\frac{p}{2\pi}$ and a constant amplitude. Solution of such differential equation is

$$x = A \sin (pt + \phi) \dots\dots\dots(3)$$

where ϕ is phase angle between the displacement x and the external force F_{ext} .

Now, differentiate equation (3) with respect to t

$$\frac{dx}{dt} = A p \cos(pt - \phi) \dots \dots \dots (4)$$

$$\text{and } \frac{d^2x}{dt^2} = -A p^2 \sin(pt - \phi) \dots \dots \dots (5)$$

Equation (2) becomes

$$\begin{aligned} & -A p^2 \sin(pt - \phi) + 2\gamma A p \cos(pt - \phi) + \omega_0^2 A \sin(pt + \phi) \\ & = F_0 \sin\{(pt - \phi) + \phi\} \\ & = F_0 [\sin(pt - \phi) \cos \phi + \cos(pt - \phi) \sin \phi] \dots \dots \dots (6) \end{aligned}$$

Now, comparing the coefficients of $\sin(pt - \phi)$ and $\cos(pt - \phi)$ of equation (6)

$$A(\omega_0^2 - p^2) = F_0 \cos \phi \dots \dots \dots (a)$$

$$2\gamma A p = F_0 \sin \phi \dots \dots \dots (b)$$

Amplitude of forced oscillation:

Squaring and adding equations (a) and (b)

$$A^2[(\omega_0^2 - p^2)^2 + 4\gamma^2 p^2] = F_0^2$$

$$A = \frac{F_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4\gamma^2 p^2}} \dots \dots \dots (6)$$

Phase difference (ϕ):

Dividing equations (b) by (a)

$$\frac{F_0 \sin \phi}{F_0 \cos \phi} = \frac{2\gamma A p}{A(\omega_0^2 - p^2)}$$

$$\phi = \tan^{-1} \left(\frac{2\gamma p}{\omega_0^2 - p^2} \right) \dots \dots \dots (7)$$

Forced angular frequency (p):

For maximum amplitude of forced oscillation, first derivative of equation (6) with respect to p is zero.

$$\text{i.e., } \frac{dA}{dp} = 0$$

$$\text{or, } 2(\omega_0^2 - p^2)(-2p) + 4\gamma^2(2p) = 0$$

$$\text{or, } (\omega_0^2 - p^2) + 2\gamma^2 = 0$$

$$\text{or, } p^2 = \omega_0^2 - 2\gamma^2$$

$$\text{or, } p = \sqrt{\omega_0^2 - 2\gamma^2} = \omega_r \text{ is called resonance frequency.}$$

Resonance:

When the frequency of applied force is equal to the natural frequency of the oscillator itself, this phenomenon is called *resonance*.

We know the amplitude of forced oscillation is

$$A = \frac{F_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4\gamma^2 p^2}}$$

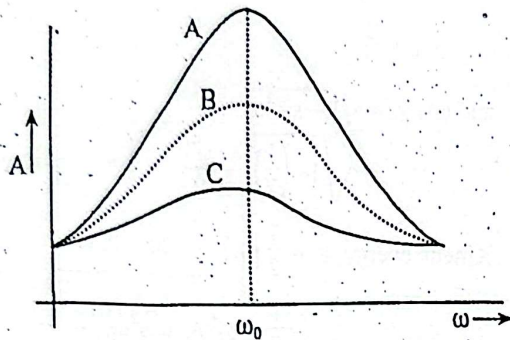
For maximum amplitude, put $p^2 = \omega_0^2 - 2\gamma^2$

$$\begin{aligned} \text{So, } A_{\max} &= \frac{F_0}{\sqrt{4\gamma^4 + 4\omega_0^2 \gamma^2 - 8\gamma^4}} \\ &= \frac{F_0}{2\gamma \sqrt{\omega_0^2 - \gamma^2}} \end{aligned}$$

At particular frequency of applied force, the amplitude of forced oscillator is maximum which is called *resonance amplitude* and the frequency is called *resonance frequency* which is given by

$$f_r = \frac{\sqrt{\omega_0^2 - 2\gamma^2}}{2\pi}$$

Amplitude of forced harmonic motion A depends on difference between applied and natural frequency. If we plot A as function of natural frequency ω , the graph will look like as shown in figure.



Curve A is the case of light damping, B is the case of heavy damping, and C is the case of over damping. The amplitude becomes large when the applied frequency ω is near the natural frequency ω_0 .

Hence, the resonance is defined as the phenomenon of setting body into vibration with its natural frequency by the application of external periodic force. Such frequency is called resonant frequency.

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SOLUTION TO EXAM PROBLEMS AND IMPORTANT QUESTIONS

1. A particle is moving with simple harmonic motion in a straight line. If it has a speed v_1 when the displacement is x_1 and speed v_2 when the displacement is x_2 then, show that the amplitude of motion is $a = \left(\frac{v_2^2 x_1^2 - v_1^2 x_2^2}{v_2^2 - v_1^2} \right)^{\frac{1}{2}}$.

⇒ Let the motion of particle be represented by

$$x = a \sin \omega t \dots\dots\dots (i)$$

$$\text{Its velocity is } v = \frac{dx}{dt} = a\omega \cos \omega t \dots\dots\dots (ii)$$

$$= a\omega \frac{\sqrt{a^2 - x^2}}{a}$$

$$= \omega \sqrt{a^2 - x^2}$$

For displacement x_1 and velocity v_1 ,

$$x_1 = a \sin \omega t$$

$$\text{and } v_1 = \omega \sqrt{a^2 - x_1^2} \dots\dots\dots (iii)$$

For displacement x_2 and velocity v_2 ,

$$v_2 = \omega \sqrt{a^2 - x_2^2} \dots\dots\dots (iv)$$

Dividing equation (iv) by (iii), we get

$$\frac{v_2}{v_1} = \frac{\sqrt{a^2 - x_2^2}}{\sqrt{a^2 - x_1^2}}$$

$$\text{or, } v_2^2 a^2 - v_2^2 x_1^2 = v_1^2 a^2 - v_1^2 x_2^2$$

$$\text{or, } a^2 (v_2^2 - v_1^2) = v_2^2 x_1^2 - v_1^2 x_2^2$$

$$\therefore a = \left(\frac{v_2^2 x_1^2 - v_1^2 x_2^2}{v_2^2 - v_1^2} \right)^{\frac{1}{2}} \text{ proved.}$$

2. An oscillatory motion of a body is represented by $y = ae^{i\omega t}$ where y is displacement in time t , a is its amplitude and ω is angular frequency. Show that the motion is simple harmonic.

⇒ The displacement of a body executing simple harmonic motion is given by

$$y = a e^{i\omega t} \dots\dots\dots(1)$$

Differentiating w.r.t. time will give

$$\frac{dy}{dt} = (i\omega) a e^{i\omega t} \dots\dots\dots(2)$$

Differentiating equation (2) will result

$$\frac{d^2y}{dt^2} = (i\omega)^2 a e^{i\omega t}$$

$$\text{or, } \frac{d^2y}{dt^2} = -\omega^2 y$$

$$\text{or, } \frac{d^2y}{dt^2} + \omega^2 y = 0 \dots\dots\dots(3)$$

which is a second order differential equation of simple harmonic motion. Hence, the motion represented by equation (1) is simple harmonic.

3. In simple harmonic motion, when the displacement is one-half the amplitude, what fraction of the total energy is KE and what fraction is PE? At what displacement is the energy half KE and half PE?

⇒ Let the simple harmonic motion of particle be represented by

$$y = A \sin \omega t \dots\dots\dots(1)$$

$$\text{Velocity of particle (v)} = \frac{dy}{dt}$$

$$= A\omega \cos \omega t \dots\dots\dots(2)$$

When displacement is half the amplitude i.e., $y = \frac{A}{2}$

$$\text{or, } \frac{A}{2} = A \sin \omega t$$

$$\text{or, } \sin \omega t = \frac{1}{2}$$

$$\begin{aligned} \text{and } \cos \omega t &= \sqrt{1 - \sin^2 \omega t} \\ &= \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2} \end{aligned}$$

$$\text{Kinetic energy, } K = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \omega^2 A^2 \cos^2 \omega t$$

$$= \frac{1}{2} m \omega^2 A^2 \frac{3}{4}$$

$$= \frac{3}{8} m \omega^2 A^2$$

$$\text{Potential energy, } U = \frac{1}{2} k y^2$$

where k is force constant and $k = m\omega^2$.

$$\text{or, } U = \frac{1}{2} k A^2 \sin^2 \omega t$$

$$= \frac{1}{2} m \omega^2 A^2 \frac{1}{4}$$

$$= \frac{1}{8} m \omega^2 A^2 \dots\dots\dots(2)$$

Total energy of the system is given by

$$E = \frac{1}{2} k A^2 = \frac{1}{2} m \omega^2 A^2 \dots\dots\dots(3)$$

$$\text{Now, } \frac{K}{E} = \frac{\frac{3}{8} m \omega^2 A^2}{\frac{1}{2} m \omega^2 A^2} = \frac{3}{4} \text{ and, } \frac{U}{E} = \frac{\frac{1}{8} m \omega^2 A^2}{\frac{1}{2} m \omega^2 A^2} = \frac{1}{4}$$

Second Part

For kinetic energy to be half of total energy,

$$\frac{K}{E} = \frac{1}{2}$$

$$\text{or, } \frac{\frac{1}{2}mv^2}{E} = \frac{1}{2}$$

$$\text{or, } \frac{\frac{3}{8}m\omega^2A^2\cos^2\omega t}{\frac{1}{2}m\omega^2A^2} = \frac{1}{2}$$

$$\text{or, } \cos^2\omega t = \frac{1}{2}$$

$$\text{and } \sin\omega t = \frac{1}{\sqrt{2}} \left[\because \sin\omega t = \sqrt{1 - \cos^2\omega t} \right]$$

$$\therefore y = A \sin\omega t = \frac{A}{\sqrt{2}}$$

Hence, at displacement $\frac{A}{\sqrt{2}}$, particle has half kinetic energy and half potential energy.

4. At $t = 0$, the displacement $x(0)$ of the block in linear oscillator is -8.50 cm. The block's velocity $v(0)$ then is 0.920 m/s and its acceleration $a(0)$ is 47 m/s².

Find: (a) angular frequency (b) phase constant

$\Rightarrow$ Here, displacement $x(0) = -8.5$ cm $= -0.085$ m

Velocity $v(0) = -0.920$ m/s

Acceleration $a(0) = 47$ m/s²

(a) Acceleration $a = -\omega^2x$

$$\text{or, } \omega^2 = -\frac{a}{x}$$

$$= -\frac{47}{(-0.085)} = 552.9$$

$\therefore$ Angular frequency $(\omega) = 23.5$ rad/s

(b) For phase constant α , $\tan\alpha = \frac{v}{\omega x}$

$$= \frac{-0.920}{23.5(-0.0850)}$$

$$= 0.461$$

$$\therefore \alpha = 25^\circ$$

5. A 0.5 kg body oscillates in SHM on a spring that, when extended 2 mm from its equilibrium position, has an 8 N restoring force. What are the angular frequency of oscillation and the capacitance of an LC circuit with the same period if inductor is 5 H.

$\Rightarrow$ Mass of body $(m) = 0.5$ kg

Extension length $(x) = 2$ mm $= 0.002$ m

Restoring force $(F) = 8$ N

Angular frequency $(\omega) = ?$

Inductance $(L) = 5$ H

Capacitance of equivalent LC oscillation $(C) = ?$

From Hook's law, $F = kx$ (ignoring negative sign)

$$\text{or, } k = \frac{F}{x}$$

$$\therefore k = \frac{8}{0.002} = 4000 \text{ Nm}^{-1}$$

$$\text{Angular frequency } (f) = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{4000}{0.5}} = 25.23 \text{ Hz}$$

$$\text{For LC oscillation, } \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

$$\text{or, } \frac{k}{m} = \frac{1}{LC}$$

$$\therefore C = \frac{m}{kL}$$

$$= \frac{0.5}{4000 \times 5} = 25 \times 10^{-6} \text{ F}$$

6. The balance wheel of watch oscillates with an angular amplitude of π radian and a period of 0.5 sec. Find

- Maximum angular speed of wheel
- The angular acceleration of wheel when the displacement is $\pi/2$ radian. [2013 Fall]

$\Rightarrow$ Angular amplitude (A) = π

Time period (T) = 0.5s

- Maximum angular speed is $v = A\omega$

$$= A \frac{2\pi}{T}$$

$$= \pi \times \frac{2\pi}{0.5}$$

$$= 4\pi^2 \text{ rad/s}$$

- Angular acceleration (α) = $\omega^2 y$

$$= \left(\frac{2\pi}{T}\right)^2 y$$

$$= \left(\frac{2\pi}{0.5}\right)^2 \times \frac{\pi}{2}$$

$$= 8\pi^3 \text{ rad/s}^2$$

7. Calculate the wavelength, frequency, speed of wave, and maximum particle velocity in the wave represented by $y = 20 \sin \pi(2t - 0.05x)$. The value of x and y are in centimeters. [2013 Spring]

$\Rightarrow$ Comparing the above equation with standard wave equation $y = A \sin(\omega t - kx)$, we get

amplitude (A) = 20 cm

angular velocity (ω) = 2π

or, $2\pi f = 2\pi$

$\therefore$ Frequency (f) = 1 Hz

The wave vector (k) = $\frac{2\pi}{\lambda}$

$$\text{or, } 0.05\pi = \frac{2\pi}{\lambda}$$

$\therefore$ Wavelength (λ) = 40 cm

Speed of wave (v) = $f\lambda = 40 \text{ cm/s}$

And the maximum particle speed ($v_{\max}$) = $A\omega = 40\pi \text{ cm/s}$

8. Calculate the frequency and maximum particle velocity due to wave represented by $y(x,t) = 0.03 \sin(60\pi t - 0.03\pi x)$. The value of x and y are in centimeters. [2013 Spring]

$\Rightarrow$ Comparing the above equation with standard wave equation

$y = A \sin(\omega t - kx)$, we get

amplitude (A) = 0.03 cm

angular velocity (ω) = 60π

or, $2\pi f = 60\pi$

Therefore, $f = 30 \text{ Hz}$

Now,

The maximum particle velocity ($v_{\max}$) = $A\omega = 0.03 \times 60\pi$
 $= 1.8\pi \text{ cm/s}$

9. Calculate the wavelength, frequency, speed of the wave and maximum particle velocity in the wave represented by $y = 10 \sin(8\pi t - 0.08\pi x)$. The value of x and y are in CGS system. [2016 Fall]

$\Rightarrow$ Comparing the above equation with standard wave equation $y = A \sin(\omega t - kx)$, we get

Amplitude (A) = 10cm

Angular velocity (ω) = 8π

or, $2\pi f = 8\pi$

$\therefore f = 4 \text{ Hz}$

The wave vector (k) = $\frac{2\pi}{\lambda} = 0.08\pi$

Therefore the wavelength (λ) = 25cm

Speed of wave (v) = $f\lambda = 100 \text{ cm/s}$

Maximum particle speed ($v_{\max}$) = $A\omega = 80\pi \text{ cm/s}$

10. What is amplitude, the wavelength and the velocity of the wave represented by $y = 5\sin(6\pi t - 4x)$ where distance and time are measured in S.I Units? [2019 Fall]

$\Rightarrow$ Comparing the above equation with standard wave equation $y = A\sin(\omega t - kx)$

Where, amplitude (A) = 5m

Angular velocity (ω) = 6π

or, $2\pi f = 6\pi$

Hence the frequency is (f) = 3Hz

The wave vector (k) = $\frac{2\pi}{\lambda} = 4$

Therefore the wavelength (λ) = $\pi/2\text{m}$

Speed of wave (v) = $f\lambda = \frac{3\pi}{2} \text{ m/s}$

11. Show that the fractional change in frequency of damped oscillation is $\frac{1}{8Q^2}$, where Q is quality factor.

$\Rightarrow$ Let f and f' be the initial and final frequency of oscillation.

Then, $f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$ (i) For free oscillation

$f' = \frac{1}{2\pi} \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ (ii) for damped oscillation

Fractional change in frequency = $\frac{f - f'}{f}$

$$= \frac{\frac{1}{2\pi} \left(\sqrt{\frac{k}{m}} - \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \right)}{\frac{1}{2\pi} \sqrt{\frac{k}{m}}}$$

$$= \sqrt{\frac{k}{m}} \left\{ \frac{1 - \sqrt{1 - \frac{b^2 m}{4m^2 k}}}{\sqrt{\frac{k}{m}}} \right\}$$

$$= 1 - \sqrt{1 - \frac{b^2}{4mk}}$$

$$= 1 - \sqrt{1 - \frac{b^2}{4m(m\omega^2)}} \quad (\because k = m\omega^2)$$

$$= 1 - \sqrt{1 - \frac{b^2}{m^2 \omega^2}}$$

$$= 1 - \sqrt{1 - \frac{1}{4Q^2}} \quad (\because Q = \frac{m\omega}{b})$$

$$= 1 - \left(1 - \frac{1}{2} \times \frac{1}{4Q^2} + \dots \right) \text{ using binomial expansion}$$

Neglecting higher terms,

$$= \frac{1}{8Q^2} \text{ proved}$$

12. If the relaxation time of a damped harmonic oscillator is 60 sec, find the time for which (a) the amplitude falls to $1/e$ times the initial value (b) the energy falls to $1/e^2$ times the initial value.

$\Rightarrow$ Relaxation time (τ) = 60 sec

Case I:

For damped harmonic oscillator,

$$\text{Amplitude (A)} = A_0 e^{-\frac{t}{2\tau}}$$

$$\text{Given, } A = \frac{A_0}{e}, t = ?$$

$$\frac{A_0}{e} = A_0 e^{-\frac{t}{2\tau}}$$

$$\text{or, } e^{-1} = e^{-\frac{t}{2\tau}}$$

$$\therefore t = 2\tau = 2 \times 60 = 120 \text{ sec}$$

Case II:

$$\text{Energy (E)} = E_0 e^{-\frac{t}{\tau}}$$

$$\text{Given } E = \frac{E_0}{e^2}, t = ?$$

$$\text{So, } \frac{E_0}{e^2} = E_0 e^{-\frac{t}{\tau}}$$

$$\text{or, } e^{-2} = e^{-\frac{t}{\tau}}$$

$$\therefore t = 2\tau = 2 \times 60 = 120 \text{ sec}$$

13. The amplitude of a lightly damped oscillator decreases by 3% during each cycle. What fraction of the energy of the oscillator is lost in each full oscillation?

⇒ Amplitude of a damped oscillator is

$$A = A_0 e^{-\frac{t}{2\tau}}$$

$$\text{Here, } A = 0.97 A_0$$

$$0.97 A_0 = A_0 e^{-\frac{t}{2\tau}} \Rightarrow t = 0.06\tau$$

Energy of a damped oscillator is

$$E = E_0 e^{-\frac{t}{\tau}}$$

$$\text{or, } \frac{E}{E_0} = e^{-\frac{0.06\tau}{\tau}} = e^{-0.06} = 0.941$$

$$\text{Fraction of the energy lost} = \frac{E_0 - E}{E_0}$$

$$= 1 - \frac{E}{E_0}$$

$$= (1 - 0.941)$$

$$= 0.058$$

$$= 0.06 \approx 6\%$$

14. A 750g block oscillates on the end of a spring whose force constant, $k = 56 \text{ N/m}$. The mass moves in a fluid which offers a resistive force $F = -bv$, where $b = 0.162 \text{ Ns/m}$. What is the period of the oscillation?

⇒ The period of oscillation is given as

$$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}$$

$$\text{Here, } k = 56 \text{ N/m, } m = 0.75 \text{ kg, } b = 0.162 \text{ Ns/m}$$

$$\therefore T = 0.7268 \text{ sec}$$

15. A meter stick is suspended as a physical pendulum through its one end. Calculate its time period and length of the equivalent simple pendulum.

⇒ We know that time period of compound pendulum is

$$T = 2\pi \sqrt{\frac{I}{mg}}$$

Moment of inertia of the stick through its end can be obtained from the parallel axis theorem, $I = I_{cm} + m\left(\frac{L}{2}\right)^2$

$$= m \frac{L^2}{12} + m \frac{L^2}{4}$$

$$= \frac{mL^2}{3}$$

$$\therefore T = 2\pi \sqrt{\frac{\frac{mL^2}{3}}{mg \frac{L}{2}}}$$

$$= 2\pi \sqrt{\frac{2L}{3g}}$$

$$= 2\pi \sqrt{\frac{2 \times 1}{3 \times 9.8}} = 1.64 \text{ s}$$

Now, for equivalent simple pendulum, time period is $2\pi \sqrt{\frac{2L}{3g}}$

$= 2\pi \sqrt{\frac{l}{g}}$ where l is length of equivalent simple pendulum,

$$\therefore l = 2 \frac{L}{3}$$

$$= 2 \times \frac{1}{3} = 0.66 \text{ m}$$

16. A mass of 1 kg is suspended from a spring of spring constant 25 N/m. if the undamped frequency is $\frac{2}{\sqrt{3}}$ times the damped frequency, what will be the damping factor?

$\Rightarrow$ Here, mass (m) = 1 kg

Spring constant (k) = 25 N/m

Damping factor (b) = ?

$$\text{Undamped frequency (f)} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \dots\dots (i)$$

$$\text{Damped frequency (f')} = \frac{1}{2\pi} \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \dots\dots (ii)$$

From given condition, $f = \frac{2}{\sqrt{3}} f'$

$$\text{or, } \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{2}{\sqrt{3}} \frac{1}{2\pi} \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Squaring and on solving, $b^2 = mk = 25 \times 1$

$$\therefore b = 5 \text{ kg/s}$$

17. In damped harmonic motion, calculate the time in which (i) its amplitude and (ii) its energy falls to $1/e$ of its undamped value if the mass of the system is 0.25 gm and damping constant is 0.01 g/s.

$$\Rightarrow \text{Relaxation time } (\tau) = \frac{m}{b}$$

$$= \frac{0.25}{0.01} = 25 \text{ s}$$

- (i) The damped amplitude is $x = x_0 e^{-\frac{t}{2\tau}}$

$$\text{or, } \frac{x_0}{e} = x_0 e^{-\frac{t}{2\tau}}$$

$$\text{or, } e^{-1} = e^{-\frac{t}{2\tau}}$$

$$\therefore t = 2\tau = 50 \text{ s}$$

- (ii) Energy of damped motion is

$$E = E_0 e^{-\frac{t}{\tau}}$$

$$\text{or, } \frac{E_0}{e} = E_0 e^{-\frac{t}{\tau}}$$

$$\text{or, } e^{-1} = e^{-\frac{t}{\tau}}$$

$$\therefore t = \tau = 25 \text{ s}$$

18. A solid sphere of mass 3kg and diameter 0.20m is suspended on a wire. Find the period of angular oscillation for small displacements if the torsional constant of the wire is $6 \times 10^{-3} \text{ Nm/rad}$.

⇒ We know that time period of torsional pendulum is

$$T = 2\pi \sqrt{\frac{I}{C}}$$

Where I is moment of inertia of solid sphere, given by

$$I = \frac{2}{5} MR^2$$

$$= \frac{2}{5} \times 3 \times (0.2)^2$$

$$= 0.048 \text{ kgm}^2$$

$$\text{Now, time period } T = 2\pi \sqrt{\frac{0.048}{6 \times 10^{-3}}}$$

$$= 2.83 \text{ second.}$$

Hence the time period of given oscillation is 2.83s.

19. A wire has a torsional constant of 2 Nm/radian. A disk of radius 5 cm and mass 100 gram is suspended through the wire. Find the frequency of such torsional oscillation.

[2022 Fall]

⇒ Torsional constant (C) = 2 Nm/rad

Mass of disc (m) = 100 gm = 0.1 kg

Radius of disc (R) = 5 cm = 0.05 m

Moment of inertia of disc (I) = $\frac{1}{2} mR^2$

$$= \frac{1}{2} \times 0.1 \times (0.05)^2$$

$$= 1.25 \times 10^{-4} \text{ kgm}^2$$

Frequency of torsional oscillation :-

$$f = \frac{1}{2\pi} \sqrt{\frac{C}{I}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{2}{1.25 \times 10^{-4}}}$$

$$= 20.13 \text{ Hz}$$

EXERCISE

1. A particle is moving with simple harmonic motion in a straight line. If it has a speed v_1 when the displacement is x_1 and speed v_2 when the displacement is x_2 then, show that the time period of motion is $2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$.
2. A meter scale is suspended as a physical pendulum through its 75 cm marks. Find its time period. *Ans: 1.517 s*
3. A simple pendulum of length l and mass m is suspended in a car that is travelling with a constant speed v around a circle of radius R . if the pendulum undergoes small oscillation about its equilibrium position x , prove its frequency is $f = \frac{1}{2\pi}\sqrt{\frac{gx}{lR}}$.
4. A metallic disc of mass 0.5kg and radius 10cm is suspended by a wire of length 40cm and radius 1mm; the torsion oscillation of the disc are found to have a period of 2.5s. Find the modulus of rigidity of the wire. *Ans: 4020 N/m²*
5. A 485g body oscillates on a spring that, when 2.1mm extended from equilibrium, has restoring force of 1.13N. (a) Calculate the frequency of oscillation (b) period of oscillation (c) what is the capacitance of analogous LC system if L is chosen to be 5.2H
Ans: 89.3 rad/s, 70.3ms, 24.1μF



CHAPTER 2

WAVE MOTION

- 2.1 Wave Motion
- 2.2 Mechanical Waves in String
 - 2.2.1 Relation Between Particle Speed and Wave Speed
- 2.3 Energy in Progressive Wave
- 2.4 Sound Waves
- 2.5 Standing Waves and Resonance
- 2.6 Sonometer

2.1 Wave Motion

Any disturbance that travels onwards through a medium due to the repeated periodic motion of its particles about their mean position is called *wave*. It may also be defined simply as the transfer of energy from one point to another. For example, we know that water waves, sound waves, light waves, and radio waves all carry energy from one point to another. For light and radio waves, propagation, no medium is required at all. There is just propagation of energy takes place. Such mode of transfer of energy is called wave motion and may be divided into two categories:

i. Mechanical wave motion:

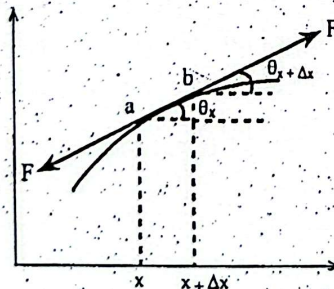
It is possible only in material media (solid, liquid, and gas) which possess inertia as well as elasticity, water waves and sound waves are example of mechanical waves.

ii. Non-mechanical or electromagnetic wave motion:

It does not require material medium for its propagation so that light and radio waves can travel through empty space.

2.2 Mechanical Waves in String

Consider a string is stretched along x-axis under tensile force F and it is displaced slightly along y-axis to generate waves in string.



Resultant force on the small segment ab of the string

$$\begin{aligned}
 &= F \sin \theta_{x+\Delta x} - F \sin \theta_x \\
 &= F \tan \theta_{x+\Delta x} - F \tan \theta_x \quad \text{for small } \theta_x \text{ and } \theta_{x+\Delta x} \\
 &= F (\tan \theta_{x+\Delta x} - \tan \theta_x) \\
 &= F \left[\left(\frac{dy}{dx} \right)_{x+\Delta x} - \left(\frac{dy}{dx} \right)_x \right] \\
 &= F \frac{d^2 y}{dx^2} \Delta x \quad \dots\dots\dots (i)
 \end{aligned}$$

Also, resultant force = ma

$$= \mu \Delta x \frac{d^2 y}{dt^2} \quad \dots\dots\dots (ii)$$

From equations (i) and (ii),

$$\mu \Delta x \frac{d^2 y}{dt^2} = F \frac{d^2 y}{dx^2} \Delta x$$

$$\text{or, } \frac{d^2 y}{dx^2} = \frac{1}{F/\mu} \frac{d^2 y}{dt^2}$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{1}{v^2} \frac{d^2 y}{dt^2}$$

where $v = \sqrt{\frac{F}{\mu}}$ is velocity of wave.

2.2.1 Relation Between Particle Speed and Wave Speed

The general equation of S.H.M. is

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

$$\text{Particle speed, } v_{\text{particle}} = \frac{dy}{dt}$$

$$= \frac{2\pi va}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x)$$

Also, slope of displacement curve is

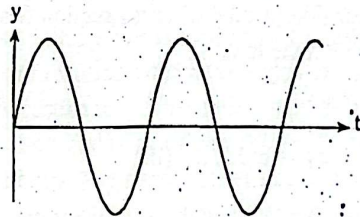
$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x)$$

$$\text{or, } \frac{dy}{dx} = -\frac{v_{\text{particle}}}{v}$$

Hence, for a wave propagated on string, the slope of displacement curve is the ratio of particle velocity to the wave velocity.

2.3 Energy in Progressive Wave

For a particle executing SHM, the velocity is maximum at equilibrium position and will be zero at extreme position. Since kinetic energy is proportional to square of velocity, it is maximum at mean position and zero at extreme position. Consequently, the potential energy at extreme position is maximum and will be zero at mean position.



To move particle from mean position to distance y , work has to be done against acceleration. For small displacement dy , work done is

$$dW = F dy$$

If ρ is density of the medium, work done per unit volume for small displacement dy is

$$\begin{aligned} dw &= m a_p dy \\ &= \rho a_p dy ; a_p \text{ is acceleration, } m = \rho \text{ for unit volume} \\ &= \rho \left[\frac{4\pi^2 a v^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \right] dy \end{aligned}$$

Total work done (w) = $\int dw$

$$= \frac{4\pi^2 \rho v^2}{\lambda^2} \int_0^y a \sin \frac{2\pi}{\lambda} (vt - x) dy$$

Work done per unit volume is stored as potential energy per unit volume u which is given by

$$u = \frac{4\pi^2 \rho v^2}{\lambda^2} \int_0^y y dy$$

$$\text{or, } u = \frac{4\pi^2 \rho v^2}{\lambda^2} \frac{y^2}{2}$$

$$\text{or, } u = \frac{4\pi^2 \rho v^2}{\lambda^2} \frac{a^2}{2} \sin^2 \frac{2\pi}{\lambda} (vt - x)$$

$$\therefore u = \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \sin^2 \frac{2\pi}{\lambda} (vt - x) \dots\dots\dots (1)$$

Kinetic energy per unit volume is

$$k = \frac{1}{2} \rho \left(\frac{dy}{dt} \right)^2 ; \rho = m \text{ for unit volume.}$$

$$= \frac{1}{2} \rho \left[\frac{2\pi a v}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \right]^2$$

$$\therefore k = \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda} (vt - x) \dots\dots\dots (2)$$

Equations (1) and (2) illustrate that the potential energy and kinetic energy of every particle will change with time.

Also,

$$\begin{aligned} (K)_{\text{avg}} &= \frac{1}{T} \int_0^T K dt \\ &= \frac{1}{T} \int_0^T K dt \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda} (vt - x) dt \\ &= \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \frac{1}{T} \int_0^T \cos^2 \frac{2\pi}{\lambda} (vt - x) dt \end{aligned}$$

$$\begin{aligned}
&= \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \frac{1}{T} \int_0^T \frac{1}{2} (1 + \cos 2(\frac{2\pi}{\lambda})(vt-x)) dt \\
&= \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \frac{1}{T} \left[\int_0^T \frac{1}{2} dt + \frac{1}{2} \int_0^T \cos \frac{4\pi}{\lambda} (vt-x) dt \right] \\
&= \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \frac{1}{T} \left[\frac{T}{2} + \left[\frac{1}{2} \frac{\sin 4\pi}{\lambda} (vt-x) \times \frac{\lambda}{4\pi v} \right]_0^T \right] \\
&= \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \frac{1}{T} \left[\frac{T}{2} + \frac{1}{2} \frac{\lambda}{4\pi v} \frac{\sin 4\pi}{\lambda} (vT-x) - \frac{\sin 4\pi}{\lambda} (0-x) \right] \\
&= \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \frac{1}{T} \left[\frac{T}{2} + \frac{\lambda}{8\pi v} \frac{\sin 4\pi}{\lambda} \left(\frac{\lambda}{T} T - x \right) + \frac{\sin 4\pi}{\lambda} x \right] \\
&= \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \frac{1}{T} \left[\frac{T}{2} - \frac{\sin 4\pi}{\lambda} + \frac{\sin 4\pi}{\lambda} \right] \\
&= \frac{\pi^2 \rho a^2 v^2}{\lambda^2} \dots \dots \dots (3)
\end{aligned}$$

Similarly,

$$(u)_{avg} = \frac{\pi^2 \rho a^2 v^2}{\lambda^2} \dots \dots \dots (4)$$

From (3) and (4), it is clear that average kinetic energy per unit volume equals to average potential energy per unit volume and is constant.

2.4 Sound Waves

Consider a sound wave travelling along +ve x-direction is

$$y = a \sin(\omega t - kx) \dots \dots (i)$$

$$\text{Velocity, } v = \frac{dy}{dt}$$

$$= a \cos(\omega t - kx), \omega = \omega a \cos(\omega t - kx)$$

Total energy of particle is

$$E = K.E + P.E.$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} k y^2 \quad [\because k = m\omega^2]$$

$$= \frac{1}{2} m \{ \omega a \cos(\omega t - kx) \}^2 + \frac{1}{2} m \omega^2 \{ a \sin(\omega t - kx) \}^2$$

$$= \frac{1}{2} m \omega^2 a^2 \{ \cos^2(\omega t - kx) + \sin^2(\omega t - kx) \}$$

$$\therefore E = \frac{1}{2} m \omega^2 a^2 \dots \dots \dots (ii)$$

If there are 'n' identical particles per unit volume, then energy density is given by,

$$U = nE$$

$$= \frac{1}{2} (nm) \omega^2 a^2$$

$$= \frac{1}{2} \rho \omega^2 a^2 \quad [\rho = nm \text{ is density}]$$

If we consider a tube of cross section S and length L, then total energy within the tube is

$$W = UV$$

$$= \frac{1}{2} \rho \omega^2 a^2 SL \dots \dots (iii)$$

where V = SL is volume of the tube.

Intensity due to wave is defined as the flow of energy per unit area per second.

$$\text{i.e., } I = \frac{W}{St}$$

$$= \frac{\frac{1}{2} \rho \omega^2 a^2 SL}{St}$$

$$= \frac{1}{2} \rho \omega^2 a^2 v \dots \dots \dots (iv) \quad [\because v = \frac{L}{t}]$$

Since the sound waves are emitted uniformly in all direction from the speaker, the expression for spherical wave is

$$I = \frac{P}{4\pi r^2} \dots\dots(v) \text{ where } P \text{ is power of sound.}$$

$$\text{or, } \frac{1}{2} \rho \omega^2 a^2 v = \frac{P}{4\pi r^2}$$

$$\text{or, } a^2 = \frac{P}{2\pi r^2 v \rho \omega^2}$$

$$\therefore a = \frac{1}{r} \sqrt{\frac{P}{2\pi \rho v \omega^2}}$$

2.5 Standing Waves and Resonance

When two progressive wave of same frequency and amplitude travelling through the medium with same speed but opposite in direction superimpose each other, they give rise to a wave called standing or stationary wave.

When a standing wave is formed, the point of maximum or zero amplitude are resulted alternatively due to the interference of wave is called resonance. The point where the amplitude is maximum are called nodes and those where the amplitude is zero are called anti-nodes. The formation of standing waves along with nodes and antinodes is shown in figure below.

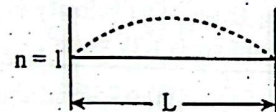


Fig.: 1

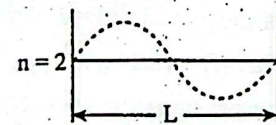


Fig.: 2

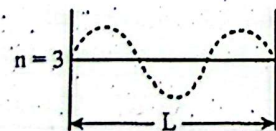


Fig.: 3

Let a string of length 'L' is fixed between two clamps and pluck up and release to produce standing wave,

In figure 1, wavelength, $\lambda = 2L$

In figure 2, wavelength, $\lambda = L$

In figure 3, wavelength, $\lambda = \frac{2L}{3}$

Continuing this process, we observe the wavelength pattern given by relation;

$$\lambda = \frac{2L}{n}, \text{ where } n = 1, 2, 3, 4, \dots\dots$$

And the resonance frequency is given by

$$f = \frac{v}{\lambda} = \frac{nv}{2L}$$

The oscillation with lowest frequency ($n=1$) is called fundamental mode or first harmonics.

Resonance in standing waves of sound occurs when the frequency of sound waves matches the natural frequency of the resonant system, resulting in a large amplitude vibration. Resonance is an important phenomenon in many areas of physics and engineering. Resonance can be used to design and optimize the performance of electrical circuits (Filters, Oscillators, Amplifiers). Resonance is also used in MRI to produce image of internal organs, blood vessels and others structures. Resonance also finds its applications in the design and optimization of performance of mechanical systems (bridge, buildings and vehicles).

2.6 Sonometer

A sonometer is an acoustic instrument consists of a wire in hollow sounding box. One end of the wire is fixed and the other is passes over a pulley and carries a load to produce tension in the wire. The vibrating length of wire can be adjusted by means of two sharp knife edges. A horse shoe magnet is placed at the middle of the wire an a.c. current is passed through the wire. A current

carrying conductor experiences a magnetic force and deflect towards the direction of force. Due to the direction of A.C. current the wire deflect upward and downward causing vibration.

The frequency of fundamental mode is,

$$f = \frac{1}{2l} \sqrt{\frac{T}{\mu}}$$

where,

$T = mg$ is tension in the wire

μ is linear mass density and l is mode of vibration.

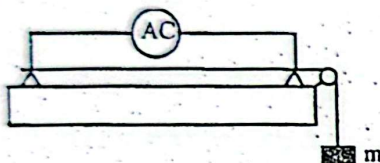


Fig.: Sonometer



SOLUTION TO EXAM PROBLEMS AND IMPORTANT QUESTIONS

1. A source of sound has a frequency 512 Hz and amplitude of 0.50 cm. Calculate the energy flow across a square of unit area per sec. The velocity of sound in air is 332 m/s and density of air is 1.29 kg/m³.

⇒ Frequency of sound (f) = 512 Hz

Amplitude of sound (A) = 0.5 cm = 0.5×10^{-2} m

Velocity of sound in air (v) = 332 ms⁻¹

Density of air (ρ) = 1.29 kg/m³

Energy flow per square unit per sec (I) = ?

We have,

$$\begin{aligned} \text{Intensity (I)} &= \frac{1}{2} \rho v \omega^2 A^2 \\ &= \frac{1}{2} \times 1.29 \times 332 \times (2\pi \times 512)^2 \times (5 \times 10^{-3})^2 \\ &= 55347 \text{ watt/m}^2 \end{aligned}$$

2. A source of sound has a frequency of 256 Hz and amplitude of 0.50 cm, calculate the energy flow across a square cm per sec. The velocity of sound in air is 330m/s and density of air is 1.29 kg/m³.

⇒ Frequency of sound (f) = 256 Hz

Amplitude of sound (A) = 0.5 cm = 0.5×10^{-2} m

Velocity of sound in air (v) = 330 ms⁻¹

Density of air (ρ) = 1.29 kg/m³

Energy flow per square unit per sec (I)

$$= \frac{1}{2} \rho v \omega^2 A^2$$

$$= \frac{1}{2} \times 1.29 \times 330 \times (2\pi \times 256)^2 \times (0.5 \times 10^{-3})^2$$

$$= 3092 \text{ J/m}^2\text{s}$$

$$\text{Energy flow across a square cm per sec} = \frac{3092}{(100 \times 100)}$$

$$= 0.3092 \text{ J/cm}^2\text{s}$$

3. A string has linear density 525 gram/m³ and tension 45N. When sinusoidal wave of frequency 120 Hz and amplitude 8.5 mm is sent along the string, at what average rate does the wave transport energy?

⇒ Here, linear density (μ) = 525 g/m

Tension (τ) = 45 N

Frequency of wave (f) = 120 Hz

Amplitude (y_m) = 8.5 mm = 8.5×10^{-3} m

$$\omega = 2\pi f = 2\pi \times 120 = 754 \text{ rad/s}$$

$$\text{Velocity of wave, } v = \sqrt{\frac{\tau}{\mu}}$$

$$= \sqrt{\frac{45}{0.525}} = 9.26 \text{ m/s}$$

$$\text{Average power transferred, } P_{av} = \frac{1}{2} \mu v \omega^2 y_m^2$$

$$\therefore P_{av} = \frac{1}{2} \times 0.525 \times 9.26 \times 754^2 \times 0.0085^2$$

$$= 100 \text{ watt}$$

4. A wave of frequency 500 Hz has a phase velocity of 200 m/s. (i) How far apart are two points 30° out of phase? (ii) What is the phase difference between two displacements at a point at times 10⁻³ s apart?

⇒ Frequency of wave (f) = 500 Hz

Phase velocity (v) = 200 m/s

$$v = \lambda f \Rightarrow \lambda = \frac{v}{f} = \frac{200}{500} = 0.4 \text{ m}$$

$$\text{Time period (T)} = \frac{1}{f}$$

$$= \frac{1}{500} = 2 \times 10^{-3} \text{ sec}$$

- i. A cycle is equivalent to 360° so that 30° corresponds to $\left(\frac{1}{12}\right)^{\text{th}}$ of a cycle.

$$\therefore \text{Corresponding length is } \frac{\lambda}{12} = \frac{0.4}{12}$$

$$= \frac{1}{30} \text{ m} = 0.033 \text{ m}$$

- ii. Here the time 10⁻³ sec is half of time period $T = 2 \times 10^{-3}$ sec and corresponds to half of one cycle or half of 360°.

Thus, the phase difference is 180°.

5. A rod vibrating at 12 Hz generates harmonic waves with amplitude of 1.5 mm in a string of linear mass density 2 gm/m. If the tension in the string is 15 N, what is the average power supplied by the source?

$$\Rightarrow \text{Velocity (v)} = \sqrt{\frac{T}{\mu}}$$

$$= \sqrt{\frac{15}{2 \times 10^{-3}}} = 61.23 \text{ m/s}$$

Average power supplied by the source is

$$P = \frac{1}{2} \mu v \omega^2 a^2$$

$$= \frac{1}{2} \times 2 \times 10^{-3} \times 61.23 \times (2\pi \times 12)^2 \times (1.5 \times 10^{-3})^2$$

$$= 0.78 \text{ Watt/m}^2$$

6. Show that the function $y = A \sin kx \cos \omega t$ satisfies the wave function.

$\Rightarrow$ Here, the function is $y = A \sin kx \cos \omega t$(i)

Differentiate with respect to t , $\frac{dy}{dt} = -A\omega \sin kx \sin \omega t$(ii)

Again differentiating, $\frac{d^2y}{dt^2} = -A\omega^2 \sin kx \cos \omega t$(iii)

$$\frac{d^2y}{dt^2} = -\omega^2 y \text{.....(iv)}$$

Again, differentiating equation (i) with respect to x twice, we get

$$\frac{d^2y}{dx^2} = -Ak^2 \sin kx \cos \omega t \text{.....(v)}$$

$$\frac{d^2y}{dx^2} = -k^2 y \text{.....(vi)}$$

From equation (iv) and (vi)

$$\frac{d^2y}{dx^2} = \frac{k^2}{\omega^2} \frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{v^2} \frac{d^2y}{dt^2} \text{.....(vii)}$$

where $v = \omega/k$ is the velocity of wave

Equation (vii) represent the wave travelling with velocity v .
Hence, the given function represents a wave function.

7. A stretched string has a linear density 525 g/m and is under tension 45N. We send a sinusoidal wave with a frequency 120Hz and amplitude 8.5mm along the string from one end. At what average rate does the wave transport energy? [2014 Spring]

$\Rightarrow$ Tension on wire (T) = 500N

Linear mass density (μ) = 525 gm/m = 0.525 kg/m

Frequency of wave (f) = 120 Hz

$$\begin{aligned} \text{Amplitude of wave (A)} &= 8.5 \text{ mm} \\ &= 0.0085 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Now, speed of wave in rope (v)} &= \sqrt{\frac{T}{\mu}} \\ &= 30.86 \text{ m/s.} \end{aligned}$$

Energy transported by wave (P)

$$\begin{aligned} &= 2\pi^2 \mu f^2 A^2 v \\ &= 2\pi^2 \times 0.525 \times (120)^2 \times (0.0085)^2 \times 30.86 \\ &= 332.72 \text{ watt.} \end{aligned}$$

8. Calculate frequency if vibration of air particles in plane progressive wave of amplitude 2.18×10^{-10} m and intensity 10^{-10} W/m², the velocity of sound in is 340 m/s and density of air is 0.00129 gm/cc ? [2015 Fall]

$\Rightarrow$ Intensity of wave (I) = 10^{-10} W/m²

Velocity of wave (v) = 340 m/s

Density of medium (ρ) = 0.00129 gm/cm³ = 1.29 kg/m³

Linear mass density (μ) = 525 gm/m = 0.525 kg/m

Amplitude of wave (A) = 2.18×10^{-10} m

Frequency of wave (f) = ?

We have,

$$\text{Intensity of progressive wave (I)} = 2\pi^2 \rho f^2 A^2 v$$

$$\text{or, } 10^{-10} = 2\pi^2 \times 1.29 \times f^2 \times (2.18 \times 10^{-10})^2 \times 340$$

$$\therefore f = 492.99 \text{ Hz}$$

9. A progressive and stationary, simple harmonic wave having frequency 250 Hz and each having same velocity 30 m/s.

i. Determine the phase difference between two vibrating points in a progressive wave at a distance of 10 cm.

ii. Wave equation of progressive wave if amplitude is 0.03 m.

iii. Distance between nodes in stationary wave.

[2015 Spring]

⇒ Velocity of wave (v) = 30 m/s

Frequency of wave (f) = 250 Hz

Now, $v = \lambda \times f$

or, $30 = \lambda \times 250$

∴ $\lambda = 0.12$ m

i. Phase difference (ϕ)

$$= \frac{2\pi x}{\lambda}$$

$$= \frac{2\pi \times 0.1}{0.12} = 5.236 \text{ rad}$$

ii. Wave equation is

$$y = 0.03 \sin(500t - 5.236)$$

iii. Distance between nodes is

$$x = \frac{\lambda}{2} = \frac{0.12}{2} = 0.06 \text{ m}$$

10. Calculate the speed of transverse wave in a rope of length 20 cm having 60 gm of wire under tension of 500N.

[2014 Fall]

⇒ Length of wire (L) = 20 cm

Mass of wire (M) = 60 gm

Tension on wire (T) = 500 N

Now, linear mass density (μ) = $\frac{M}{L}$

$$= \frac{60}{20}$$

$$= 3 \text{ gm/cm}$$

$$\begin{aligned} \text{Speed of wave in rope } (v) &= \sqrt{\frac{T}{\mu}} \\ &= \sqrt{\frac{500}{3}} = 12.91 \text{ cm/s.} \end{aligned}$$

11. A transverse sinusoidal wave is generated at one end of a long horizontal string by a bar, which moves up and down through a distance of 0.5m. The motion is continuous and repeated regularly twice each second. If the string has linear mass density of 0.005kg/m and is kept under tension of 2N. Find the speed, amplitude, time period and wavelength of the wave motion.

[2016 Spring]

⇒ Amplitude of wave (A) = 0.5m

Time period (T) = 0.5s

Linear mass density (μ) = 0.005 kg/m

Tension on wire (F) = 20N

$$\begin{aligned} \text{Now, speed of wave in rope } (v) &= \sqrt{\frac{F}{\mu}} \\ &= \sqrt{\frac{20}{0.005}} \\ &= 63.24 \text{ m/s} \end{aligned}$$

$$\begin{aligned} \text{And wavelength } (\lambda) &= v \times f = \frac{v}{T} \\ &= \frac{63.24}{0.5} \\ &= 126.48 \text{ m} \end{aligned}$$

12. A sinusoidal wave travels along a string. The time for a particular point to move from maximum displacement to zero is 0.17s.

i. What are the period and frequency?

ii. The wavelength is 1.40 m, what is the wave speed?

[2017 Fall]

$$\Rightarrow \text{i. Time period (T)} = 4 \times 0.17$$

$$= 0.68 \text{ s}$$

$$\text{Hence the frequency is (f)} = \frac{1}{T}$$

$$= 1.47 \text{ Hz}$$

$$\text{ii. Speed of wave (v)} = f\lambda$$

$$= 1.47 \times 1.40 = 2.059 \text{ m/s}$$

13. Two progressive waves of equal amplitude and frequency travelling in opposite direction superimpose to each other so as to form a standing wave of equation $y = A \cos kx \sin \omega t$, where $A = 1 \text{ mm}$, $k = 1.57 \text{ cm}^{-1}$ and $\omega = 78.5 \text{ sec}^{-1}$. Find:

- Velocity of progressive wave
- Node closet to the origin, $x > 0$
- Antinodes closet to the origin, $x > 0$
- Amplitude of resultant wave when, $x = 2.33 \text{ cm}$

[2017 Spring]

$$\Rightarrow \text{Amplitude of wave (A)} = 1 \text{ mm} = 0.001 \text{ m}$$

$$\text{Wave vector (k)} = 1.57 \text{ cm}^{-1} = 157 \text{ m}^{-1}$$

$$\text{Angular frequency (\omega)} = 78.5 \text{ s}^{-1}$$

$$\text{We have wavelength of wave (\lambda)} = \frac{2\pi}{k}$$

$$= \frac{2\pi}{157} = 0.04 \text{ m}$$

$$\text{i. Velocity of wave (v)} = \frac{\omega}{k}$$

$$= 0.5 \text{ m/s}$$

$$\text{ii. Node closet to origin} = \frac{\lambda}{2}$$

$$= 0.02 \text{ m}$$

$$\text{iii. Antinode closet to origin} = \frac{\lambda}{4}$$

$$= 0.01 \text{ m}$$

$$\text{iv. Amplitude at given point is (y_0)}$$

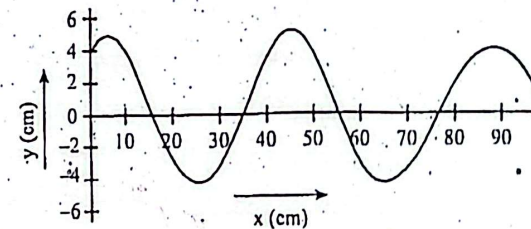
$$= A \cos kx$$

$$= 0.001 \cos (157 \times 0.0233)$$

$$= 9.98 \times 10^{-4} \text{ m} = 0.99 \text{ mm}$$

14. A simple harmonic transverse wave is propagating along a string towards the left direction as shown in figure. Figure shows a plot of displacement as a function of position, at time $t=0$. The string tension is 3.6 N and its linear density is 25 g/m . Calculate:

- The amplitude
- The wavelength
- Wave speed
- The period
- The maximum particle speed in the string.
- Write an equation describing the travelling wave.



[2018 Fall]

- $$\Rightarrow \text{(i) Amplitude of the wave from figure is } 5 \text{ cm.}$$
- $$\text{(ii) Wavelength of the wave from figure is } 40 \text{ cm.}$$
- $$\text{(iii) Wave speed (v)}$$

$$= \sqrt{\frac{F}{\mu}}$$

$$= \sqrt{\frac{3.6}{0.025}} = 12 \text{ m/s.}$$

(iv) Time period (T)

$$= \frac{\lambda}{v}$$
$$= \frac{0.4}{12} = 0.033 \text{ sec.}$$

(v) Maximum particle speed ($v_{\max}$)

$$= A\omega = A \frac{2\pi}{T}$$
$$= 0.05 \times \frac{2\pi}{0.033} = 9.52 \text{ m/s.}$$

(vi) Equation of the travelling wave is $y = 0.05 \sin 5\pi(9.52t - x)$

15. A stretched string has a linear mass density of 5 gm/cm and a tension of 10 N. A wave on the string has amplitude of 0.12 mm and a frequency of 100 Hz is travelling in negative x-direction.

i. Write the wave equation with appropriate units.

ii. At what average rate does the wave transport energy?
[2018 Spring]

$\Rightarrow$ Tension on wire (T) = 10 N

Linear mass density (μ) = 5 gm/cm = 0.5 kg/m

Frequency of wave (f) = 100 Hz

Amplitude of wave (A) = 0.12 mm = 1.2×10^{-4} m

Now, speed of wave in rope (v) = $\sqrt{\frac{T}{\mu}}$

$$= \sqrt{\frac{10}{0.5}} = 4.472 \text{ m/s.}$$

Energy transported by wave (P)

$$= 2\pi^2 \mu f^2 A^2 v$$
$$= 2\pi^2 \times 0.5 (100)^2 \times (1.2 \times 10^{-4})^2 \times 4.472$$
$$= 6.35 \times 10^{-3} \text{ watt}$$

16. A piano wire with mass 5gm and length 90cm is stretched with tension of 25N. a wave with frequency 100Hz and amplitude 1.6mm travels along the wire; a) Calculate the average power carried by the wave b) What happens to the power if the wave amplitude is halved.
[2020 Fall]

$\Rightarrow$ Tension on wire (T) = 25N

Linear mass density (μ) = 5gm/90cm = 0.055g/cm = 5.5kg/m

Frequency of wave (f) = 100Hz

Amplitude of wave (A) = 1.6mm = 0.0016m

Now, speed of wave in rope (v) = $\sqrt{\frac{T}{\mu}}$

$$= 2.13 \text{ m/s.}$$

(a) We have, energy transport by wave (P) = $2\pi^2 \mu f^2 A^2 v$

$$= 5.92 \text{ watt.}$$

(b) If the wave amplitude of wave is halved power of the wave will be (P') = $2\pi^2 \mu f^2 \left(\frac{A}{2}\right)^2 v$

$$= \frac{P}{4}$$

Hence the power of the wave will reduced by one fourth.

17. Here are the three equations of wave:

$$y(x,t) = 2\sin(4x-2t), y(x,t) = 2\sin(3x-4t), y(x,t) = 2\sin(3x-3t)$$

Rank the waves according to their (a) wave speed and (b) maximum transverse speed with greatest first.

[2021 Fall]

$\Rightarrow$ Comparing above equation with standard wave equation,
 $y = A \sin(\omega t - kx)$

For first wave $\omega = 4$ and $k = 2$

$$\text{Wave speed } (v_1) = \frac{\omega}{k} = \frac{4}{2} = 2 \text{ m/s}$$

And the maximum transverse speed $(v_{\max}) = A\omega = 2 \times 4$
 $= 8 \text{ m/s}$

For second wave $\omega = 3$ and $k = 4$

$$\text{Wave speed } (v_2) = \frac{\omega}{k}$$

$$= \frac{3}{4} = 0.75 \text{ m/s}$$

And the maximum transverse speed $(v_{\max}) = A\omega = 2 \times 3$
 $= 6 \text{ m/s}$

For third wave $\omega = 3$ and $k = 3$

$$\text{Wave speed } (v_3) = \frac{\omega}{k}$$

$$= \frac{3}{3} = 1 \text{ m/s}$$

And the maximum transverse speed $(v_{\max}) = A\omega = 2 \times 3$
 $= 6 \text{ m/s}$

Hence rank of wave according to their wave speed is $y(x, t) = 2\sin(4x - 2t)$, $y(x, t) = 2\sin(3x - 3t)$, $y(x, t) = 2\sin(3x - 4t)$

Maximum transverse speed of first one is maximum and remaining two are same.

18. A wave is propagating on a long stretched string along its length taken as the positive x-axis. The wave equation is given by $y = y_0 e^{-\left(\frac{t}{T} - \frac{x}{\lambda}\right)^2}$ where $y_0 = 4 \text{ mm}$, $T = 1 \text{ s}$ and $\lambda = 4 \text{ cm}$.

- Find the velocity of wave
- Find the function $f(t)$ giving the displacement of the particle at $x = 0$.
- Find the function $g(x)$ giving the shape of the string at $t = 0$.
- Plot the shape $g(x)$ of the string at $t = 0$
- Plot the shape of the string at $t = 5 \text{ sec}$. [2021 Fall]

$\Rightarrow$ Here, the wave equation for given value of amplitude, time period and wavelength in CGS unit is,

$$y = 0.4 e^{-\left(\frac{t}{1} - \frac{x}{4}\right)^2}$$

$$y = 0.4 e^{-(t - 0.25x)^2}$$

- i. Velocity of wave is

$$v = \frac{dy}{dt}$$

$$= 0.4 e^{-(t - 0.25x)^2} \times (-2)(t - 0.25x)$$

$$\therefore v = -0.8(t - 0.25x) e^{-(t - 0.25x)^2}$$

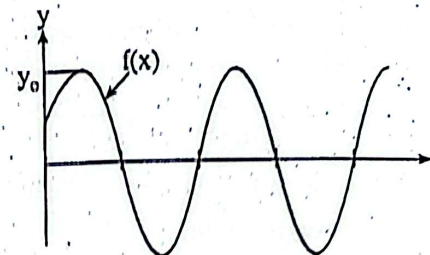
- ii. The function giving the displacement of the particle at x

$$: \text{ at } x = 0 \text{ is } f(t) = 0.4 e^{-t^2}$$

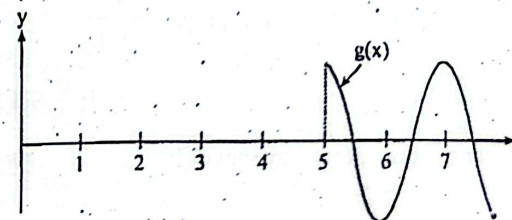
- iii. Function giving the shape of the string at $t = 0$ is

$$g(x) = 0.4 e^{-0.06x^2}$$

- iv. shape of string at $t = 0$ is



- v. shape of string at $t = 5 \text{ sec}$ is



19. A string of mass 20 gram and length 1.6 m is fixed on a rigid support at one end. The other end is attached with a mass of 2 kg. Calculate the speed of transverse pulse formed on the string. [2022 Fall]

$$\Rightarrow \text{Mass of string (M)} = 20 \text{ gm} = 20 \times 10^{-3} \text{ kg}$$

$$\text{Length of string (L)} = 1.6 \text{ m}$$

$$\text{Linear mass density } (\mu) = \frac{M}{L}$$

$$= \frac{20 \times 10^{-3}}{1.6}$$

$$= 1.25 \times 10^{-2} \text{ kg/m}$$

$$\text{Tension on string (T)} = mg$$

$$= 2 \times 9.8$$

$$= 20.6 \text{ N}$$

$$\begin{aligned} \text{Speed of transverse pulse (v)} &= \sqrt{\frac{T}{\mu}} \\ &= \sqrt{\frac{20.6}{1.25 \times 10^{-2}}} \\ &= 0.406 \text{ m/s} \end{aligned}$$

EXERCISE

1. A stretched spring has a linear density 525g/m, and is under tension 45N. A sinusoidal wave of frequency 120Hz and amplitude 0.85cm is sent from one end. At what average rate does the wave transport energy? *Ans: 100W*
2. A uniform rope of mass m and length L hangs from a ceiling.
 - a. Show that the speed of a transverse wave on the rope is a function of y, the distance from the lower end is given by $v = \sqrt{gy}$.
 - b. Show that the time a transverse wave takes to travel the length of the rope is given by $t = 2\sqrt{\frac{L}{g}}$.

❖❖❖

- 3.1 Introduction
 - 3.1.1 Sound Friendly Auditorium Hall
- 3.2 Sabine's Relation
- 3.3 Ultrasound
 - 3.3.1 Production of Ultrasonic Wave - Piezoelectric Generation
- 3.4 Ultrasonic Method in Non-Destructive Testing
- 3.5 Noise Pollution

3.1 Introduction

Acoustics is the branch of physics that deals with the study of all mechanical waves in solid, liquid and gases medium including vibration sound, ultra sound and infra sound ect. Acoustic phenomena can be discussed in term of frequency into three sections: *audio*, *ultrasonic*, and *infrasonic*. The range of *audio wave* falls between 20Hz to 20 KHz which can be detected by human ear and has number of applications in speech communication and music. *Ultrasonic waves* refer to waves of frequency higher than 20 KHz and are useful in imaging technology such as ultrasonography and elastography. The lowest frequency wave is called *infrasonic* which is used to study geological phenomena such as earthquakes.

3.1.1 Sound Friendly Auditorium Hall

For the clear and distinct sound from the speaker, the hall should have the following characteristics:

a. **Structural design**

Distribution of intensity throughout the hall should be uniform. Usual design of the hall should be parabolic shape. At the speaker's end as shown each syllable of sound should be distributed in such a way that each word is heard distinctly and there is no reinforcement causing the change in quality. Coved wall or corners should be bare minimum because there may be unduly concentration of sound at some place and some other place may become zone of silence.

b. **Materials to be used**

In friendly auditorium, there should be provision to absorb the unnecessary reflected sound. Some porous material like clothes, asbestos, cushions, etc. should be used and they are

put at various places of the hall. Open window is considered as perfect absorber of sound since there is no reflection and sound simply passes through. So, to get optimum value of reverberation time, we should have few windows.

c. Reverberation

Interval of time taken by continuous sound to fall to an intensity by 60 decibels in loudness is called reverberation. Fall intensity of sound in hall is exponential. So, it will take longer time to become zero. Due to multiple reflection of wall, ceiling and floor sound reverberates and persist in side room for longer time. For good acoustic, reverberation time should have optimum value (not too small not too large). If small, sound vanishes instantaneously and gives the hall dead effort. If too large, there is multiple reflection and overlapping thereby causing confusion. Different frequency of sound may interfere differently at some point. So, quality of sound may change. This produces unpleasant effect, especially in time of music.

These are some suggestions to make sound friendly auditorium hall.

3.2 Sabine's Relation

The reverberation time of sound can be obtained from the volume, surface area and absorption coefficient of the sound absorber in hall by simple relation known as *Sabine's relation*.

Let I be the intensity of sound in the room at any instant, δI be the fall in intensity in small interval of time δt , n is the number of reflections per second. It has been found that,

$$\delta I \propto I$$

$$\delta I \propto n$$

$$\delta I \propto \delta t$$

Combining these,

$$\delta I \propto n I \delta t$$

$$\text{or, } \delta I = -\alpha n I \delta t$$

where α is mean absorbing coefficient and $n = \frac{3v}{4V}$. The negative sign indicates the decrease of intensity.

In the expression of n , S is total surface area, V is volume of room, and v is velocity of sound.

Now, equation (1) can be expressed as

$$\delta I = -\alpha \frac{Sv}{4V} I \delta t$$

$$\text{or, } \frac{\delta I}{I} = -\alpha \frac{Sv}{4V} \delta t \quad \dots\dots\dots(2)$$

Integrating equation (2), we have

$$\int_{I_0}^I \frac{\delta I}{I} = -\alpha \frac{Sv}{4V} \int_0^t \delta t$$

$$\text{or, } [\ln I]_{I_0}^I = -\alpha \frac{Sv}{4V} [t]_0^t$$

$$\text{or, } \ln I - \ln I_0 = -\alpha \frac{Sv}{4V} (t - 0)$$

$$\text{or, } \ln \left(\frac{I}{I_0} \right) = -\alpha \frac{Sv}{4V} t$$

$$\text{or, } \frac{I}{I_0} = e^{-\alpha \frac{Sv}{4V} t}$$

From the definition of reverberation time T , $I = \frac{I_0}{10^6}$

$$10^{-6} = e^{-\alpha \frac{Sv}{4V} T}$$

$$\text{or, } 6 \ln 10 = \alpha \frac{Sv}{4V} T$$

$$\text{or, } T = \frac{6 \ln 10 \times 4V}{\alpha Sv}$$

Taking velocity of sound, $v = 332 \text{ m/s}$

$$T = \frac{0.166V}{\alpha S} \text{ which is Sabine's relation.}$$

Hence, the reverberation time decreases with increasing the sound absorbing area.

3.3 Ultrasound

The sound wave which has frequency greater than 20,000 Hz is called *ultrasonic sound wave*. Since they travel with velocity of sound, marginal wavelength of such wave is $\frac{33200}{20000} = 1.66$ cm at room temperature.

3.3.1 Production of Ultrasonic Wave - Piezoelectric Generation

Certain crystals like quartz, tourmaline, etc. exhibit electrical charges when heated or cooled. This phenomenon is called *piezoelectricity*.

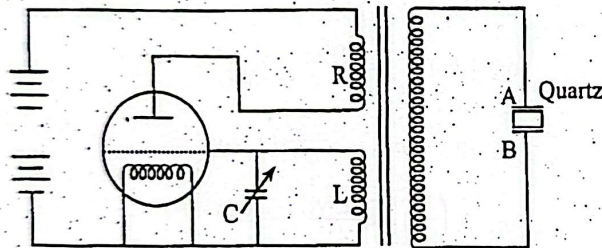


Figure shows a simple piezoelectric generator. Q is a slice of crystal placed between metal plates A and B so as to form parallel plate condenser. The plates are connected to the primary of a transformer which is coupled inductively to the oscillator circuit of a valve as shown in figure. If natural frequency of oscillation of valve circuit is equal to the frequency of the crystal mode to which we intend to excite, resonance will occur and crystal will undergo linear expansion and contraction at exactly same rate. With moderate size crystal, ultrasonics of frequency 5,40,000 cycles/sec can be produced. These vibrations are longitudinal in nature and frequency of vibration is given by

$$f = \frac{k}{2l} \sqrt{\frac{Y}{\rho}}$$

where $k = 1, 2, 3, \dots$ etc. is mode of vibration

Y = Young's modulus of elasticity

ρ = density of the crystal

Ultra sound	Ordinary sound
1. Frequency of ultrasound is greater than 20,000 Hz	1. Frequency of ordinary sound ranges from 20 Hz to 20,000 Hz.
2. It can't be heard.	2. It can be heard.
3. It has many applications in engineering.	3. It has limited applications.

Applications of ultrasonic wave:

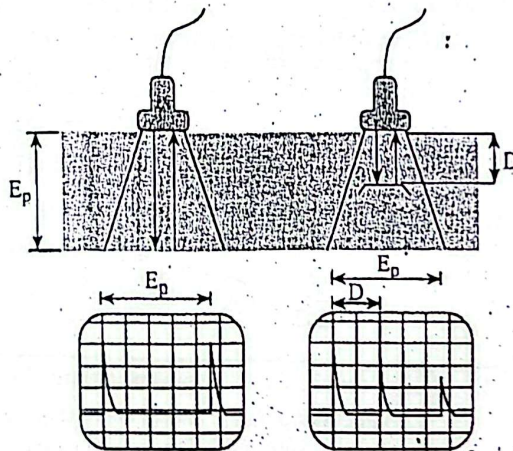
1. Ultrasonic waves are use to diagnosed different disease and monitor different organs inside the human body in medical field.
2. Ultrasonic waves are used in navigation purpose in SONAR.
3. Ultrasonic waves are use in research and survey, to measure distances and positions.
4. Ultrasonic waves are used in industry to find cracks and defects.
5. Ultrasonic waves are used as catalytic agent in chemical reactions.
6. Ultrasonic waves are used to cut and welding and soldering metals.
7. Ultrasonic waves are used in cleaning and cavitation purpose.
8. Ultrasonic waves are used in environmental monitoring and quality control in manufacturing.

3.4 Ultrasonic Method in Non-Destructive Testing

Ultrasonics is the name given to the study and application of ultrasound. A *non-destructive test* is one in which there is no impairment of the properties and performance in future use of the object under examination. The ultrasonic method of non-destructive testing can be used for flaw detection, dimensional measurements, material characterization, etc. The ultrasonic method of non-destructive testing is currently the most widely used of the testing methods. The ultrasonic method is used in many industries including steel and aluminium construction, metallurgy, manufacturing, aerospace, automotive, and other transportation sectors.

Working principle:

In most common ultrasonic method of testing, a very short ultrasonic pulse waves with frequencies ranging from 0.1-15 MHz and sometimes up to 50 MHz are transmitted into materials under test.



As shown in left figure, a probe sends ultrasonic wave into a test material. There are two indications shown in the cathode ray oscilloscope (CRO): one from the initial pulse of the probe, and the second due to the back wall echo.

However, if there is a defect (right figure), there is a third indication that simultaneously reduces the amplitude of the back wall indication. The depth of the defect is given by the ratio $\frac{D}{E_p}$.

Advantages of ultrasonic testing:

- Testing can be carried out from a single surface.
- A high degree of penetration is possible in many commonly-used materials.
- Accuracy in locating and measuring defects.
- Ability to detect and size very small defects.
- Compatibility with automatic scanning devices and with microprocessors and computers.

Disadvantages of ultrasonic testing:

- Operators must be properly trained and highly experienced.
- Cast iron and other coarse grained materials are difficult to inspect due to low sound transmission and high signal noise.
- Linear defects oriented parallel to the sound beam may go undetected.

3.5 Noise Pollution

Any unwanted sound to human ear or other living beings is called *noise pollution*. Noise pollution causes many serious health problems to human body especially to ear and mental issues. According to WHO standard, sound above 120dB loudness is considered as noise pollution. Sound with such high amplitude cause vibration to surrounding objects and may cause damage.

Causes of noise pollution:

- **Industrialization:** In industries, use of heavy machinery such as generators, motors, mills, huge exhaust fans has produced unwanted noise.

- **Vehicles:** As the number of vehicles is increasing day by day, it is also another reason of noise pollution.
- **Events:** Weddings, public gatherings, musical programs has also contributed to noise pollution.
- **Construction sites:** Mining for materials and construction of buildings, bridges, etc. are also another causes of noise pollution.

Effect of noise pollution:

1. Noise pollution causes ear problems and in long term, it may cause hearing loss.
2. Noise pollution causes various mental health problems such as anxiety, depression, stress, etc.
3. Noise pollution near soft building and structure causes damage.
4. Noise pollution near school and colleges disturbs academic activities.

Prevention:

1. Industries and factory should be far from village/city areas.
2. Proper sound insulation should be maintained in machinery devices.
3. Loudness of sound speaker should be controlled in public area.
4. Awareness about the effect of sound pollution should be given to the students from school level.



SOLUTION TO EXAM PROBLEMS AND IMPORTANT QUESTIONS

1. A room has dimensions $6\text{m} \times 4\text{m} \times 5\text{m}$. Find:
 - i. Mean free path of sound wave in the room
 - ii. The number of reflections made per second by the sound wave with the walls of the room. (Take velocity of sound in air $= 350\text{ms}^{-1}$).

$$\Rightarrow \text{Volume of room (V)} = 6\text{m} \times 4\text{m} \times 5\text{m} \\ = 120\text{m}^3$$

- i. Mean free path is the average distance covered by a sound wave through air between two consecutive reflections with the walls of the room.

$$\text{i.e., } \sigma = \frac{4 \times \text{volume}}{\text{Area}}$$

$$\begin{aligned} \text{Total surface area} &= 2(lb + bh + lh) \\ &= 2(6 \times 4 + 4 \times 5 + 6 \times 5) \\ &= 148\text{m}^2 \end{aligned}$$

$$\therefore \sigma = \frac{4 \times 120}{148} = 3.242 \text{ m}$$

- ii. Number of reflections made per second by sound wave is

$$\begin{aligned} n &= \frac{\text{velocity}}{\text{mean free path}} \\ &= \frac{350}{3.243} = 108 \end{aligned}$$

2. How much acoustic power enters the window of area 1.58 m^2 , through the sound wave (standard intensity level 10^{-16} W/cm^2)? The window opens on a street where the street noise results in an intensity level at the window of 60 dB.

⇒ Area of window (A) = 1.58 m²

$$\begin{aligned}\text{Standard intensity } (I_0) &= 10^{-16} \text{ W/cm}^2 \\ &= 10^{-12} \text{ W/m}^2\end{aligned}$$

Intensity level = 60 dB

We know that, intensity level = $10 \log \left(\frac{I}{I_0} \right)$

$$\text{or, } 60 = 10 \log \frac{I}{10^{-12}}$$

$$\therefore I = 10^{-6} \text{ W/m}^2$$

$$\begin{aligned}\text{Acoustic power} &= \text{Intensity} \times \text{Area} \\ &= 10^{-6} \times 1.58 \\ &= 1.58 \times 10^{-6} \text{ watt}\end{aligned}$$

3. The reverberation time for an empty hall is 1.5 sec. With 500 audiences present in the hall, the reverberation time falls to 1.4 sec. Find the number of persons present in the hall if the reverberation time falls down to 1.312 sec.

⇒ Reverberation time for empty hall is given by

$$T = \frac{0.165 V}{\Sigma \alpha ds} \quad \dots\dots\dots (i)$$

where V is volume of the hall and $\Sigma \alpha ds$ is the sound absorbed by the surface.

$$\text{or, } 1.5 = \frac{0.165 V}{\Sigma \alpha ds}$$

When 500 audiences are present,

$$T = \frac{0.165 V}{\Sigma \alpha ds + 500\alpha} \quad \dots\dots\dots (ii)$$

where α is absorption coefficient for a single person and its value is nearly equal to 0.43.

$$\text{or, } 1.4 = \frac{0.165 V}{\Sigma \alpha ds + 500 \times 0.43} \quad \dots\dots\dots (iii)$$

Solving equation (ii) and (iii), we get $\Sigma \alpha ds = 3010 \text{ m}^2$ sabine and $V = 27363.636 \text{ m}^3$

For reverberation time reduced to 1.312 sec, let's suppose that n be the number of audiences present in hall.

$$T = \frac{0.165 V}{\Sigma \alpha ds + n \alpha}$$

$$\text{or, } 1.312 = \frac{0.165 V}{\Sigma \alpha ds + n \times 0.43}$$

Putting value of V and $\Sigma \alpha ds$, we get

$$\therefore n = 1003$$

4. The volume of a small hall in BICC is 900 m³. The wall area of the hall is 150 m², ceiling area 90 m² and floor area is 95 m². The sound absorption coefficient for wall is 0.04, for ceiling is 0.04 and for the floor is 0.08. Calculate average sound absorption coefficient and reverberation time.

⇒ Volume of hall (V) = 900 m³

$$\text{Wall area } (s_1) = 150 \text{ m}^2$$

$$\text{Ceiling area } (s_2) = 90 \text{ m}^2$$

$$\text{Floor area } (s_3) = 95 \text{ m}^2$$

$$\text{Absorption coefficient for wall } (\alpha_1) = 0.04$$

$$\text{Absorption coefficient for ceiling } (\alpha_2) = 0.04$$

$$\text{Absorption coefficient for floor } (\alpha_3) = 0.08$$

$$\text{Average sound absorption coefficient } \alpha_{av} = ?$$

$$\text{Reverberation time } (T) = ?$$

We have,

Average sound absorption coefficient

$$\alpha_{av} = \frac{\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3}{s_1 + s_2 + s_3}$$

$$= \frac{150 \times 0.04 + 90 \times 0.04 + 95 \times 0.08}{150 + 90 + 95}$$

$$= 0.05134$$

Reverberation time is

$$T = \frac{0.158 V}{\Sigma \alpha s}$$

$$= \frac{0.158 \times 900}{150 \times 0.04 + 90 \times 0.04 + 95 \times 0.08}$$

$$= 8.26 \text{ sec}$$

5. A source of sound has a frequency of 256 Hz and amplitude of 0.50 cm, calculate the energy flow across a square cm per sec. The velocity of sound in air is 330 m/s and density of air is 1.29 kg/m³.

⇒ Frequency of sound (f) = 256 Hz
 Amplitude of sound (A) = 0.5 cm = 0.5×10^{-2} m
 Velocity of sound in air (v) = 330 ms⁻¹
 Density of air (ρ) = 1.29 kg/m³
 Energy flow per square unit per sec (I) = $\frac{1}{2} \rho v \omega^2 A^2$

$$= \frac{1}{2} \times 1.29 \times 330 \times (2\pi \times 256)^2 \times (0.5 \times 10^{-2})^2$$

$$= 3092 \text{ J/m}^2\text{s}$$

Energy flow across a square cm per sec = $\frac{3092}{(100 \times 100)}$

$$= 0.3092 \text{ J/cm}^2\text{s}.$$

6. A reverberation time of 2.3 sec is observed in a hall of volume 5500 m³. The sound absorbing surface of the hall has an area of 750 m². Calculate the average absorption coefficient.

⇒ Reverberation time (T) = 2.3 s

Volume of hall (V) = 5500 m³

Surface area (S) = 750 m²

Average absorption coefficient (α) = ?

We know,

$$T = \frac{0.158 V}{\alpha S}$$

or, $\alpha = \frac{0.158 \times 5500}{2.3 \times 750} = 0.504$

Hence, the sound absorption coefficient of the hall is 0.504.

7. By how much would intensity level at a given place change when intensity of sound produced by a source at that place is doubled?

⇒ The intensity level of sound is expressed as

$$I_L = 10 \log_{10} \left(\frac{I}{I_0} \right)$$

We may write for two cases as

$$I_{L1} = 10 \log_{10} \left(\frac{I_1}{I_0} \right)$$

$$I_{L2} = 10 \log_{10} \left(\frac{I_2}{I_0} \right)$$

By the question, $I_2 = 2I_1$

$$I_{L1} = 10 \log_{10} \left(\frac{I_1}{I_0} \right)$$

$$I_{L2} = 10 \log_{10} \left(\frac{2I_1}{I_0} \right)$$

Change in intensity level is calculated as

$$I_{L2} - I_{L1} = 10 \left[\log_{10} \left(\frac{2I_1}{I_0} \right) - \log_{10} \left(\frac{I_1}{I_0} \right) \right]$$

$$= 10 \log_{10} \left(\frac{2I_1}{I_1} \right) \quad [\because \log a - \log b = \log \frac{a}{b}]$$

$$= 10 \log_{10} 2$$

$$= 3.01 \text{ dB}$$

8. The volume of a hall is 475m^3 , the area of wall is 200cm^2 , area of floor and ceiling each is 100cm^2 . If absorption coefficient of the wall, ceiling and floor are 0.03, 0.04 and 0.05 respectively, find the reverberation time for the hall.

$$\Rightarrow \text{Here, Reverberation time } (T) = \frac{0.15V}{\alpha S}$$

$$T = \frac{0.158 \times 475}{(200 \times 10^{-4} \times 0.03 + 100 \times 10^{-4} \times 0.05 + 100 \times 10^{-4} \times 0.04)}$$

$$T = 0.0005 \text{ s}$$

9. The time of reverberation for an empty hall without and with 500 audiences is 1.5 s and 1.4 s respectively. Find the reverberation time with 800 audiences in the hall.

[2013 Fall]

$$\Rightarrow \text{Reverberation time } (T) = \frac{0.158V}{\alpha S}$$

$$1.5 = \frac{0.158V}{\alpha S} \dots\dots\dots (i)$$

$$1.4 = \frac{0.158V}{\alpha S + 500} \dots\dots\dots (ii)$$

$$T = \frac{0.158V}{\alpha S + N} \dots\dots\dots (iii)$$

Solving equations (i) and (ii), we get

$$\alpha S = 7000 \text{ and } 0.158V = 10500$$

Now equation (iii) becomes,

$$T = \frac{10500}{7000 + 800} = 1.346 \text{ s}$$

10. The size of an empty assembly of bell has dimension $20 \times 15 \times 5 \text{ cm}^3$ and the reverberation time is 3.5 s. what area of the wall should be covered by curtain cloth to reduced the reverberation time by 2.5s if the absorption coefficient of curtain cloth is 0.5. Also calculate the average absorption coefficient of the bell. [2014 Fall]

$$\Rightarrow \text{Volume of the hall } (V) = 20 \times 15 \times 5 \text{ cm}^3$$

$$= 1500 \text{ cm}^3$$

$$= 1.5 \times 10^{-3} \text{ m}^3$$

$$\text{Reverberation time for empty hall } (T) = 3.5 \text{ s}$$

$$\text{Reverberation time with curtain } (T') = 2.5 \text{ s}$$

$$\text{Absorption coefficient of curtain } (\alpha_c) = 0.5$$

$$\text{Total surface area of the room } (S) = 2(lb + lh + bh)$$

$$= 950 \text{ cm}^2$$

$$= 9.5 \times 10^{-2} \text{ m}^2$$

We know,

$$\text{Reverberation time without curtain } (T) = \frac{0.158V}{\alpha S}$$

$$\text{or, } 3.5 = \frac{0.158V}{2\alpha(lb + lh + bh)}$$

$$\alpha = \frac{0.158V}{7(lb + lh + bh)}$$

$$= \frac{0.158 \times 1.5 \times 10^{-3}}{7(20 \times 15 + 15 \times 5 + 20 \times 5) \times 10^{-4}}$$

$$\therefore \alpha = 0.00026$$

Let A be the area covered by curtain cloths

$$\text{Reverberation time with curtain } (T') = \frac{0.158V}{\alpha(S-A) + A\alpha_c}$$

$$\text{or, } 2.5 = \frac{0.158 \times 1.5 \times 10^{-3}}{0.00026(9.5 \times 10^{-2} - A) + A \times 0.5}$$

$$\therefore A = 14.027 \times 10^{-5} \text{ m}^2$$

11. A classroom has dimension $20 \times 15 \times 5 \text{ m}^3$. The reverberation time is 3.5 sec. Calculate the total absorption of its surface and the average absorption coefficient. [2019 Fall]

$$\Rightarrow \text{Volume of the hall} = 20 \times 15 \times 5 \text{ cm}^3$$

$$= 1500 \text{ cm}^3$$

$$= 1.5 \times 10^{-3} \text{ m}^3$$

Reverberation time for empty hall (T) = 3.5s

$$\text{Total surface area of the room } (S) = 2(lb + lh + bh)$$

$$= 2(20 \times 15 + 15 \times 5 + 20 \times 5)$$

$$= 950 \text{ cm}^2 = 0.095 \text{ m}^2$$

$$\text{We know, Reverberation time } (T) = \frac{0.158V}{\alpha S}$$

$$\text{or, } 3.5 = \frac{0.158 \times 1.5 \times 10^{-3}}{0.095 \alpha}$$

$$\therefore \text{Average absorption coefficient } (\alpha) = 0.005$$

$$\text{Now, total absorption of the surface } (\alpha S) = 0.005 \times 0.095$$

$$= 0.00047 \text{ m}^2$$

12. A hall has volume 2265 m^3 . Its total absorption is equivalent to 92.9 m^2 of open window. What will be the effects of reverberation time if an audience fills the hall and thereby increase the absorption by another 92.8 m^2 ? [2019 Spring]

$$\Rightarrow \text{Reverberation time without audience } (T) = \frac{0.158V}{\alpha S}$$

$$= \frac{0.158 \times 2265}{92.9}$$

$$= 3.85 \text{ s}$$

$$\text{Reverberation time with audience } (T) = \frac{0.158 \times 2265}{185.8}$$

$$= 1.93 \text{ s}$$

Hence, the reverberation time will decrease from 3.85s to 1.93s if an audience fills the hall.

13. A boy claps his hands once every second and hears the echo from a distant building. He hears the echo of each clap mid-way between it and the next clap. If the velocity of sound is 340 m/s , what is the distance of the building from the body? [2019 Spring]

$$\Rightarrow \text{Velocity of sound } (v) = 340 \text{ m/s}$$

$$\text{Time taken by echo } (t) = 0.5 \text{ s}$$

Let distance of building is x .

$$\text{We have, } vt = 2x$$

$$\text{or, } 340 \times 0.5 = 2x$$

$$\therefore x = 85 \text{ m.}$$

14. For an empty assembly hall of size $20 \times 15 \times 10 \text{ m}^3$, the reverberation time is 3.5 sec. Calculate the average absorption coefficient of the hall. What area of the wall should be covered by the curtain so as to reduce the reverberation to 2.5 sec? Given the absorption coefficient of curtain cloth is 0.5. [2021 Fall]

$$\Rightarrow \text{Volume of the hall } (V) = 20 \times 15 \times 10 \text{ cm}^3$$

$$= 3000 \text{ cm}^3 = 3 \times 10^{-3} \text{ m}^3$$

$$\text{Reverberation time for empty hall } (T) = 3.5 \text{ s}$$

$$\text{Reverberation time with curtain } (T') = 2.5 \text{ s}$$

$$\text{Absorption coefficient of curtain } (\alpha_c) = 0.5$$

$$\text{Total surface area of the room } (S) = 2(lb + lh + bh)$$

$$= 2(20 \times 15 + 20 \times 10 + 15 \times 10)$$

$$= 1300 \text{ cm}^2$$

$$= 0.13 \text{ m}^2$$

We know,

$$\text{Reverberation time without curtain } (T) = \frac{0.158V}{\alpha S}$$

$$\text{or, } 3.5 = \frac{0.158V}{0.13\alpha}$$

$$\alpha = \frac{0.158 \times 3 \times 10^{-3}}{0.13 \times 3.5} = 0.00104$$

Hence average absorption coefficient of the hall is 0.00104.

Let A be the area covered by curtain cloths

$$\text{Reverberation time with curtain (T')} = \frac{0.158V}{\alpha(S-A) + A\alpha_c}$$

$$\text{or, } 2.5 = \frac{0.158 \times 3 \times 10^{-3}}{0.00104(0.13-A) + A \times 0.5}$$

$$\therefore A = 2.7 \times 10^{-4} \text{ m}^2$$

15. The size of an empty assembly hall has dimensions $20 \times 15 \times 5 \text{ m}^3$ and the reverberation time is 3.5 sec. what area of wall should be covered by curtain cloth to reduce the reverberation time by 2.5 sec; if the absorption coefficient of the curtain cloth is 0.5. Also calculate the average absorption of the hall. [2023 Spring]

$$\Rightarrow \text{Volume of the hall (V)} = 20 \times 15 \times 5 \text{ m}^3 \\ = 0.2 \times 0.15 \times 0.05 \\ = 0.0015 \text{ m}^3$$

$$\text{Total surface area of the room (S)} = 2(lb + lh + bh) \\ = 2(0.2 \times 0.15 + 0.2 \times 0.05 + 0.15 \times 0.05) \\ = 0.095 \text{ m}^2$$

Reverberation time for empty hall (T) = 3.5 s

Reverberation time with curtain (T') = 2.5 s

Absorption coefficient of curtain (α_c) = 0.5

We know the reverberation time from Sabine relation,

$$\text{Reverberation time without curtain (T)} = \frac{0.158V}{\alpha S}$$

$$\text{or, } 3.5 = \frac{0.158 \times 0.0015}{0.095\alpha}$$

$$\alpha = \frac{0.158 \times 0.0015}{0.095 \times 3.5} = 0.0007$$

Hence average absorption coefficient of the hall is 0.0007.

Let A be the area covered by curtain cloths

$$\text{Reverberation time with curtain (T')} = \frac{0.158V}{\alpha(S-A) + A\alpha_c}$$

$$\text{or, } 2.5 = \frac{0.158 \times 0.0015}{0.0007(0.095-A) + A \times 0.5}$$

$$\therefore A = 1.4 \times 10^{-4} \text{ m}^2$$

$$0.0007 \times 0.095 - 0.0007A + A \times 0.5 = \frac{2.37 \times 10^{-4}}{2.5}$$

$$5.65 \times 10^{-5} + A(0.5 - 0.0007) = 9.48 \times 10^{-5}$$

$$A(0.4993) = 2.83 \times 10^{-5}$$

$$A = 5.66 \times 10^{-5}$$

$$4.336 \times 10^{-5}$$

EXERCISE

1. A loud speaker produces sound level of 8 dB above a certain reference level at a point 40 m from it. Find the intensity level at a point 30 m from the source. *Ans: 10.5 dB*
2. A reverberation time of 2.3 s is observed in a hall of volume 5500 m^3 . The sound absorbing surface of the hall has an area of 750 m^2 . Calculate the average absorption coefficient.

Ans: 1.85

CHAPTER 4

PHOTONICS

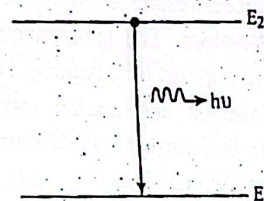
- 4.1 Laser
 - 4.1.1 Spontaneous Emission
 - 4.1.2 Stimulated Emission
 - 4.1.3 Population Inversion
 - 4.1.4 Active Medium
 - 4.1.5 Pumping
 - 4.1.6 He-Ne Laser
 - 4.1.7 Semiconductor Laser
 - 4.1.8 Applications of Laser
- 4.2 Optical Fiber
 - 4.2.1 Acceptance Angle
 - 4.2.2 Theory of Optical Fibre
 - 4.2.3 Numerical Aperture (NA)
- 4.3 Types of Optical Fibre

Chapter 4 Photonics

4.1 LASER

LASER is an optical device which produces highly intense, monochromatic, unidirectional, coherent beam of light. The term 'LASER' stands for light amplification by stimulated emission of radiation. Laser works on the principle of stimulated emission of radiation.

4.1.1 Spontaneous Emission

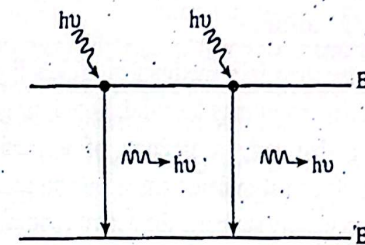


When an atom in excited state E_2 goes to the ground state E_1 by spontaneously emitting a photon, the process is known as *spontaneous emission*, and the frequency of photon is given by

$$\nu = \frac{E_2 - E_1}{h}$$

In such a case, the emitted photon can move in any random direction. Photons emitted from various atoms in the assembly have no phase relationship between themselves and thus, the radiation given out is incoherent. Example: ordinary light.

4.1.2 Stimulated Emission



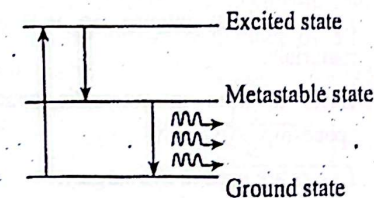
When an atom in an excited state E_2 interacts with an incident photon of frequency ν and is thereby induced to move to the ground state E_1 by emitting the difference of energy $E_2 - E_1$ as the photon of the same frequency, the process is known as *stimulated emission*. In such a case;

- the emitted photon travels in the same direction in which the incident photon is moving and
- the two photons are in phase with each other.

4.1.3 Population Inversion

The establishment of a situation in which the number of atoms in the higher energy level is greater than that in lower energy level is called *population inversion*.

Laser works on the principle of stimulated emission which is possible only when the electrons jump from higher level to lower level by some trigger. In normal state, number of atoms in lower energy level is higher than that in higher level. So, continuous transition from high level to low level is not possible.



Population inversion creates the number of atoms in higher state more so it is possible to jump these excited atoms to

ground state by emitting radiation. Hence, the laser action cannot occur without population inversion between atomic levels.

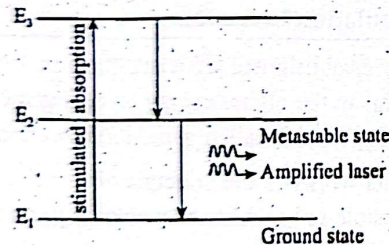
4.1.4 Active Medium

Active medium is a medium in which light gets amplified and this medium may be solid, liquid, or gas. It is important to note that only a fraction of a particular medium is responsible to stimulated emission and remaining part of the medium merely supports the active centres.

4.1.5 Pumping

The procedure adopted to achieve population inversion is called *pumping*.

In optical pumping, active medium is illuminated by light of suitable frequency $\nu = \frac{E_2 - E_1}{h}$. As a result, atoms in lower energy state E_1 absorb incident photon of energy $h\nu$ and raised to higher energy state E_2 .

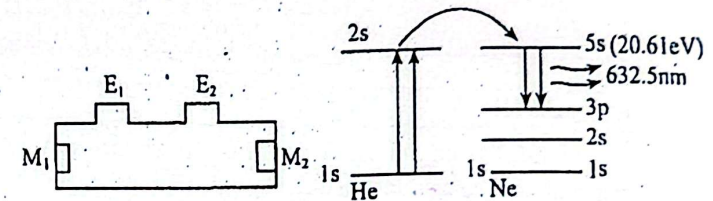


Study of laser is very important in various engineering fields. Laser is used for the following purposes:

1. Distance measurement.
2. Laser is highly intense. So, it is used in welding of materials.
3. Laser is used in automatic control of rocket and spaceship.
4. Lasers are used in eye surgery.

4.1.6 He-Ne Laser

It consists of a long narrow discharge tube filled with mixture of Helium and Neon. Two electrodes E_1 and E_2 are fitted to discharge tube. M_1 & M_2 are two mirror which forms resonant cavity.

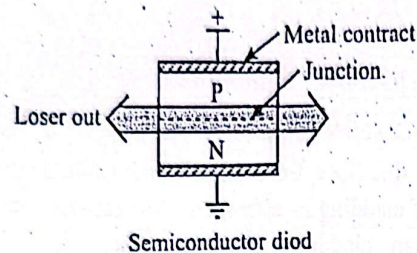


When electric field is applied, He-atoms are excited to 2s state same as energy of 5s of Ne atom. So, energy of He-atoms are transferred to Ne atoms due to the collision.

This process results in population inversion between 5s and 3p state in Ne-atom. The spontaneous transition from 5s to 3p would produce photons of wavelength 632.5nm, which then trigger stimulated emission and photon travelling parallel to tube are reflected back and forth between mirror and rapidly build up an intense beam which escape through the end with lower reflectivity, which is a laser beam.

4.1.7 Semiconductor Laser

A semiconductor laser is heavily doped PN junction diode which works on forward bias. Semiconductor diodes are highly efficient and used in optical communication. Emission of light in semiconductor is due to the recombination of electrons and holes when electric field is applied in the given material of solid light is produced. This phenomenon is called electroluminescence. The light produced by this method is incoherent as the current across pN junction increases spontaneous emission starts, increasing the current to a finite value spontaneous emission stops and the light becomes coherent.



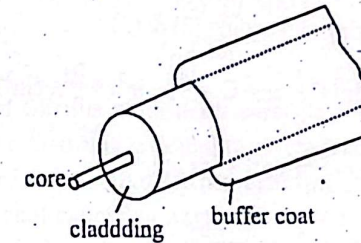
4.1.8 Applications of Laser

1. Laser lights are highly used in medical field for bloodless surgery like stone surgery, eye surgery etc.
2. Distance measurement.
3. Laser is highly intense. So, it is used in welding of materials.
4. Laser is used in automatic control of rocket and spaceship.
5. Laser is used in military to target missiles.
6. Laser is used in research field.
7. Laser is used in measuring very small time pulse such as femto second and auto seconds.
8. Laser is used in measuring very short distance.
9. Holography and optical fiber communications are result of laser.
10. Laser is used in nuclear power plant to carry thermo-nuclear reactions.
11. Laser is used in industries to cut and drill very hard metals.

4.2 Optical Fiber

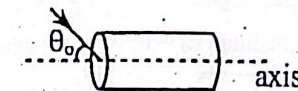
Optical fiber is a flexible optically transparent fiber, usually made of glass or plastic, through which light can be transmitted by successive total internal reflections. Optical fiber consists of three parts namely protective layer, cladding, and core. The refractive index of the core is greater than that of the cladding.

It is made by arranging concentric cylinders of different refractive index. The innermost cylinder known as *core* is made of material of very high refractive index due to the fact that it is highly doped. The cylinder outside the core is known as *cladding* whose refractive index is less than that of core because this region is lightly doped. The next layer is known as *buffer coating* (also known as *protective layer*) as shown in figure.



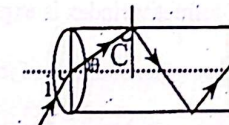
4.2.1 Acceptance Angle

Acceptance angle is defined as the maximum angle that a light ray can have relative to the axis of fiber and propagate down the fibre. The light rays contained within the cone having a full angle $2\theta_0$ are accepted and transmitted along the fibre. The cone is called acceptance cone.



4.2.2 Theory of Optical Fibre

Consider a signal is incident on an optical fibre by making an angle i with the axis of fibre, θ be the angle of refraction and C is the critical angle in core-cladding boundary. Let n_1 , n_2 , n_3 be the refractive indices of air, core and cladding respectively.



Using Snell's law at interface of air and core

$$n_0 \sin i = n_1 \sin \theta \dots\dots\dots(1)$$

For core cladding interface

$$n_1 \sin C = n_2 \sin 90^\circ \dots\dots\dots(2)$$

$$\text{From (1), } \frac{n_0}{n_1} \sin i = \sin \theta$$

$$= \sin (90^\circ - C) = \cos C$$

$$\therefore \cos C = \frac{n_0}{n_1} \sin i \dots\dots(3)$$

Squaring and adding (2) & (3)

$$\sin^2 C + \cos^2 C = \frac{n_0^2}{n_1^2} \sin^2 i + \frac{n_2^2}{n_1^2} \sin^2 90^\circ$$

$$\text{or, } 1 = \frac{n_0^2}{n_1^2} \sin^2 i + \frac{n_2^2}{n_1^2}$$

$$\text{or, } \frac{n_0^2}{n_1^2} \sin^2 i = \frac{n_1^2 - n_2^2}{n_1^2}$$

$$\text{or, } \sin^2 i = \frac{n_1^2 - n_2^2}{n_0^2}$$

Hence, $i = \sin^{-1} \left(\sqrt{\frac{n_1^2 - n_2^2}{n_0^2}} \right)$ is the required expression for acceptance angle.

For air medium $n_0 = 1$.

$$\sin i = \sqrt{n_1^2 - n_2^2}$$

4.2.3 Numerical Aperture (NA)

It is the light gathering ability of the fibre and it measures the amount of light accepted by the fibre. Numerical aperture is given by,

$$NA = \sqrt{n_1^2 - n_2^2}$$

Fractional refractive index is expressed as

$$\Delta = \frac{n_1 - n_2}{n_1}$$

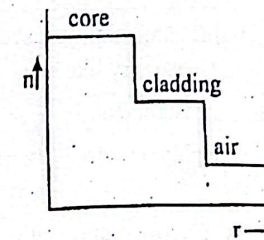
$$= \frac{n_1^2 - n_2^2}{n_1^2 (n_1 + n_2)} \approx \frac{(NA)^2}{2n_1^2}$$

$$\therefore NA = n_1 \sqrt{2\Delta}$$

4.3 Types of Optical Fibre

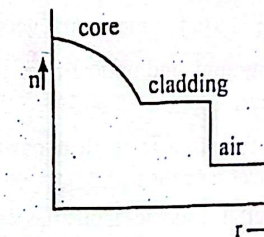
(i) Step Index Optical Fiber

In this fiber, the refractive index of the core is fixed and that of cladding is also fixed. Refractive index of core is higher than cladding. Therefore, there is noticeable boundary between the core and the cladding as shown in figure.



(ii) Graded Index Optical Fiber

In this fiber, refractive index of core is a function of radial distance from the centre of the fiber i.e., refractive index of fiber decreases smoothly from the middle to the outer surface as shown in figure.



In graded index fiber, there is low transmission loss due to self focusing.



SOLUTION TO EXAM PROBLEMS AND IMPORTANT QUESTIONS

1. An optical fiber has a NA of 0.2 and a cladding refractive index of 1.59. Determine acceptance angle for the fiber in water which has a refractive index of 1.33.

⇒ Numerical aperture (NA) = 0.2

Cladding refractive index (n_2) = 1.59

Acceptance angle (θ) = ?

Refractive index (μ) = 1.33

$$\text{We have, } NA = \frac{\sqrt{n_1^2 - n_2^2}}{n_0} \dots\dots\dots (i)$$

When fibre is in air, $n_0 = 1$ and

$$NA = \sqrt{(n_1)^2 - (n_2)^2} = 0.2$$

$$\text{or, } n_1 = \sqrt{(0.2)^2 + 1.59^2} = 1.6025$$

When the fibre is in water, $n_0 = 1.33$

$$NA = \frac{\sqrt{n_1^2 - n_2^2}}{n_0} \\ = \frac{\sqrt{1.6025^2 - 1.59^2}}{1.33} = 0.15$$

$$\text{Acceptance angle } \theta_{(max)} = \sin^{-1}(NA) \\ = \sin^{-1}(0.15) = 8.6^\circ$$

2. If the numerical aperture be 0.2441 and refractive index of core be 1.5, calculate the refractive index of the cladding and the acceptance angle in an optical fiber.

⇒ Here, Numerical aperture (NA) = 0.2441

Refractive index of core (n_1) = 1.5

Refractive index of cladding (n_2) = ?

Acceptance angle (θ) = ?

$$\text{We have } NA = \sqrt{n_1^2 - n_2^2}$$

$$n_2 = \sqrt{n_1^2 - NA^2} = 1.48$$

$$\begin{aligned} \text{Now, Acceptance angle } \theta &= \sin^{-1}(\sqrt{n_1^2 - n_2^2}) \\ &= \sin^{-1}(NA) \\ &= \sin^{-1}(0.2441) \\ &= 14.13^\circ \end{aligned}$$

3. A glass clad fibre is made with core glass of refractive index 1.5 and cladding is doped to give a fractional index difference of 0.005. Find (i) the cladding index (ii) the critical internal reflection angle (iii) the external critical acceptance angle (iv) numerical aperture (v) acceptance angle.

⇒ Refractive index of core (n_1) = 1.5

Refractive index of cladding (n_2) = ?

$$\Delta = \frac{n_1 - n_2}{n_1}$$

$$\text{or, } 0.005 \times 1.5 = 1.5 - n_2$$

$$\text{or, } n_2 = 1.4925$$

ii. Critical internal reflection angle is

$$\begin{aligned} \sin C &= \frac{n_2}{n_1} \\ &= \left(\frac{1.4925}{1.5} \right) \end{aligned}$$

$$\begin{aligned} \text{or, } C &= \sin^{-1}\left(\frac{1.4925}{1.5}\right) \\ &= 84.3^\circ \end{aligned}$$

iii. External critical acceptance angle is given by

$$n_0 \sin C = n_1 \sin (90 - 84.3)$$

$$\sin C = \frac{n_1}{n_0} \sin 5.7$$

$$\text{or, } C = \sin^{-1}(1.5 \sin 84.7) \quad (\because n_0 = 1 \text{ for air})$$

$$\therefore C = 8.56^\circ$$

$$\begin{aligned} \text{iv. } NA &= n_1 \sqrt{2\Delta} \\ &= 1.5 \sqrt{2 \times 0.005} \\ &= 0.15 \end{aligned}$$

$$\begin{aligned} \text{v. Acceptance angle, } i &= \sin^{-1}(\sqrt{n_1^2 - n_2^2}) \\ &= \sin^{-1}(\sqrt{1.5^2 - 1.4925^2}) \\ &= 8.6^\circ \end{aligned}$$

4. The dispersive powers of crown and flint glasses are 0.016 and 0.032 respectively. Calculate the focal lengths of the lenses made of crown and flint glasses which forms an achromatic doublet of equivalent focal length 20cm when placed in contact? [2014 Fall]

⇒ Equivalent focal length (f) = 20 cm

$$\omega_1 = 0.016, \omega_2 = 0.032$$

For achromatic doublet,

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0$$

$$\text{or, } \frac{0.016}{f_1} + \frac{0.032}{f_2} = 0$$

$$\text{or, } f_2 = -2f_1$$

Also,

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f}$$

$$\text{or, } \frac{1}{f_1} + \frac{1}{-2f_1} = \frac{1}{20}$$

$$\therefore f_1 = 10 \text{ cm}$$

$$f_2 = -2f_1$$

$$= -2 \times 10 = -20 \text{ cm}$$

Hence, by combining a convex lens of focal length of 10cm with a concave lens of focal length -20cm, we can design an achromatic doublet.

5. An optical fiber has fractional refractive index change of 0.02 and a cladding refractive index of 1.59. Determine acceptance angle for the fiber in water which has a refractive index of 1.33. Also evaluate Numerical aperture (NA). [2023 Spring]

⇒ Fractional refractive index change (Δ) = 0.02

Cladding refractive index (n_2) = 1.59

Acceptance angle (θ) = ?

Numerical aperture (NA) = ?

Refractive index (μ) = 1.33

We have,

$$NA = \sqrt{(n_1^2)^2 - (n_2)^2} = 0.02$$

$$\begin{aligned} n_1 &= \sqrt{(n_2)^2 + (NA)^2} \\ &= 1.59 \sqrt{1.59^2 + 0.02^2} \\ &= 1.5901 \end{aligned}$$

$$\begin{aligned} NA &= n_1 \sqrt{2\Delta} \\ &= 1.5901 \sqrt{2 \times 0.02} \\ &= 0.318 \end{aligned}$$

$$\begin{aligned} \text{Now, Acceptance angle } (\theta) &= \sin^{-1}\left(\frac{NA}{n_0}\right) \\ &= \sin^{-1}\left(\frac{0.318}{1.33}\right) \\ &= 13.83^\circ \end{aligned}$$

EXERCISE

1. Calculate the NA, acceptance angle and the critical angle of fiber having core and cladding refractive indices 1.50 and 1.45
Ans: (0.387, 22.78°, 75.2°)
2. Calculate the refractive indices of core and cladding materials of the fibre from the following data. NA = 0.22, Δ = 0.012
Ans: 1.42, 1.40
3. If the numerical aperture be 0.2441 and refractive index of core be 1.5. Calculate the refractive index of cladding and the acceptance angle.
Ans: 1.48, 14.13°

♦ ♦ ♦

CHAPTER 5

CAPACITOR AND DIELECTRIC

- 5.1 Capacitor
 - 5.1.1 General Method of Calculating Capacitance
- 5.2 Types of Capacitor
- 5.3 Charging and Discharging of Capacitor
- 5.4 Gauss Law
- 5.5 Relation Between Electric Field and Potential
- 5.6 Capacitor with Dielectric
- 5.7 Dielectrics
- 5.8 Clausius-Mossotti Equation

5.1 Capacitor

Capacitors are electric devices which store electrical charges or energy in an electric field. When a capacitor is connected with a source of potential difference 'V' then charge stored by capacitor at any time is directly proportional to V,

$$\text{i.e., } q \propto V$$

$$\text{or, } q = CV$$

where C is a constant of geometry and known as capacitance of capacitor. Its SI unit is coulomb per volt or Farad.

5.1.1 General Method of Calculating Capacitance

General method to calculate capacitance of a capacitor is as follows:

- i. Assume a charge q on the plates.
- ii. Calculate the electric field $\vec{E}$ between the plates in terms of this charge using Gauss' law i.e.

$$\oint \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0}$$

Here, q is the charge enclosed by a gaussian surface.

- iii. Calculate the potential difference V between the plates by using equation

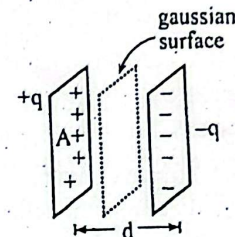
$$V = \int_+^- \vec{E} \cdot d\vec{r}$$

- iv. Calculate capacitance C using equation

$$C = \frac{q}{V}$$

5.2 Types of Capacitor

i. Parallel Plate Capacitor



Consider two parallel plates of area A, and stored charge +q and -q separated by a distance d.

Using Gauss' law,

$$\oint \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0}$$

$$\text{or, } EA = \frac{q}{\epsilon_0}$$

$$\text{or, } E = \frac{q}{A\epsilon_0}$$

Potential difference between the plates is

$$V = Ed$$

$$\text{or, } V = \frac{qd}{A\epsilon_0}$$

Now, capacitance can be calculated as

$$\text{Capacitance (C)} = \frac{q}{V}$$

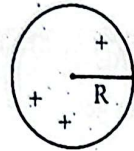
$$\text{or, } C = \frac{qA\epsilon_0}{qd}$$

$$\therefore C = \frac{\epsilon_0 A}{d} \text{ which is the required expression.}$$

ii. Spherical Capacitor

a. Capacitance of an Isolated Sphere

Consider a spherical capacitor of radius R having total charge +q.



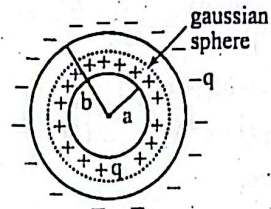
Potential at the surface of sphere is

$$V = \frac{q}{4\pi\epsilon_0 R}$$

$$\text{Capacitance (C)} = \frac{q}{V}$$

$$\therefore C = \frac{q}{\frac{q}{4\pi\epsilon_0 R}} = 4\pi\epsilon_0 R$$

b. Capacitance of Two Concentric Spheres



Consider two spheres of radii a and b having charges $+q$ and $-q$ respectively.

Using Gauss' law,

$$\oint \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0}$$

$$E (4\pi r^2) = \frac{q}{4\pi\epsilon_0 r^2}$$

Electric potential between the spheres is

$$V = \int \vec{E} \cdot d\vec{r} \\ = \frac{q}{4\pi\epsilon_0} \int_a^b \frac{1}{r^2} dr$$

$$= \frac{q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_a^b$$

$$= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{a} - \frac{1}{b} \right]$$

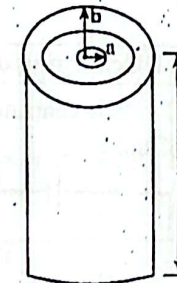
$$= \frac{q(b-a)}{4\pi\epsilon_0 ab}$$

Now, capacitance can be calculated as

$$\text{Capacitance (C)} = \frac{q}{V} \\ = \frac{4\pi\epsilon_0 ab}{b-a}$$

iii. Cylindrical Capacitor

Figure represents a section of a cylindrical capacitor of length l , inner radius a , and outer radius b .



The inner cylinder is a solid rod carrying a charge $-q$ uniformly distributed over its surface and outer conductor is a co-axial cylindrical shell carrying charge $+q$ which is also uniformly distributed. Consider a cylinder of length l and radius r as a gaussian surface.

From Gauss' law,

$$\oint \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0}$$

$$\text{or, } E (2\pi r l) = \frac{q}{\epsilon_0}$$

$$\text{or, } E = \frac{q}{2\pi\epsilon_0 l r} \dots\dots\dots (1)$$

Hence, the potential difference between a and b is

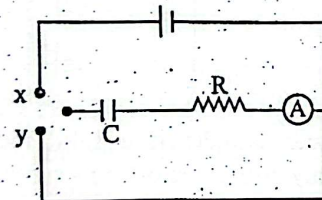
$$V = V_{ab} = \int_a^b \vec{E} \cdot d\vec{r} \dots\dots\dots (2)$$

Using equation 1 in equation 2, we get

$$\begin{aligned} V &= \frac{q}{2\pi\epsilon_0 l r} \int_a^b \frac{dr}{r} \\ &= \frac{q}{2\pi\epsilon_0 l r} \ln\left(\frac{b}{a}\right) \dots\dots\dots (3) \\ C &= \frac{q}{V} \\ &= 2\pi\epsilon_0 \frac{l}{\ln\left(\frac{b}{a}\right)} \end{aligned}$$

5.3 Charging and Discharging of Capacitor

Consider a DC circuit containing R and C in series with ammeter and source of emf E.



Growth of Charge (Charging)

When the switch x is closed, capacitor starts charging and hence, it causes growth of charge.

Applying Kirchhoff's loop rule

$$E = V_C + V_R = \frac{q}{C} + IR$$

$$\text{or, } R \frac{dq}{dt} + \frac{q}{C} = E \quad \left[\because i = \frac{dq}{dt} \right]$$

$$\text{or, } \frac{dq}{EC - q} = \frac{dt}{RC} \dots\dots\dots (1)$$

Integrating, we get

$$\int \frac{dq}{q_0 - q} = \int \frac{1}{RC} dt \quad \left[\because q_0 = EC \right]$$

$$\text{or, } -\ln(q_0 - q) = \frac{t}{RC} + K$$

where K is integrating constant.

For t = 0, q = 0,

$$K = -\ln q_0$$

$$\text{or, } -\ln(q_0 - q) + \ln q_0 = \frac{t}{CR}$$

$$\text{or, } \ln\left(\frac{q_0 - q}{q_0}\right) = -\frac{t}{CR}$$

$$\text{or, } q = q_0(1 - e^{-t/CR})$$

$$q = q_0(1 - e^{-t/\tau}) \dots\dots\dots (2)$$

Where,

$\tau = CR$ is called capacitive time constant

This is the expression for growth of charge.

At time t = τ equation (2) becomes,

$$q = q_0(1 - e^{-1}) = 0.63q_0 = 63\% \text{ of } q_0$$

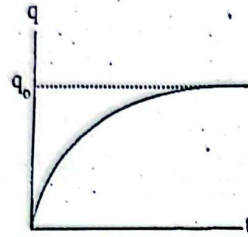
Hence the time constant (τ) of the circuit is defined as time taken to reach the charge in capacitor to 63% of maximum value.

To find the current in circuit differentiate above equation with respect to time,

$$\frac{dq}{dt} = \frac{q_0}{RC} e^{-t/RC} = I_0 e^{-t/RC}$$

$$\text{i.e., } i = I_0 e^{-t/RC} \dots\dots\dots (3)$$

This is the expression for rise of current.



Decay of Charge (Discharging)

When the capacitor is fully charged and the key is on, discharging occurs through resistor.

Again, using Kirchhoff's rule

$$0 = V_C + V_R = IR + \frac{q}{C}$$

$$\text{or, } R \frac{dq}{dt} = -\frac{q}{C}$$

$$\text{or, } \frac{dq}{q} = -\frac{1}{RC} dt$$

$$\text{Integrating, } \log q = -\frac{t}{RC} + K$$

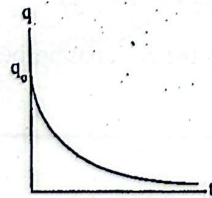
where $K = \log q_0$ is integrating constant

$$\text{or, } \ln q - \ln q_0 = -\frac{t}{RC}$$

$$\ln\left(\frac{q}{q_0}\right) = -\frac{t}{RC}$$

$$\therefore q = q_0 e^{-t/RC}$$

This is the required expression for decay of charge.



To find the capacitance of capacitor, we differentiate above equation with respect to time.

$$\frac{dq}{dt} = -\frac{q_0}{RC} e^{-t/RC} = -I_0 e^{-t/RC}$$

$$I = -\frac{dq}{dt}$$

$$\therefore I = I_0 e^{-t/RC}$$

If $t_{1/2}$ is the time for current reduced to half of maximum, then

$$\frac{I_0}{2} = I_0 e^{-t_{1/2}/RC}$$

Taking log on both sides, we get

$$\text{or, } \frac{t_{1/2}}{RC} = \ln 2$$

$$\text{or, } C = \frac{t_{1/2}}{R \ln 2}$$

$$\therefore C = \frac{t_{1/2}}{0.693R}$$

Noting the value of $t_{1/2}$ and R , capacitance of capacitor can be found.

5.4 Gauss Law

It states that the total electric flux through a closed surface enclosing a charge is equal to $\frac{1}{\epsilon_0}$ times the total charge enclosed by the surface. Mathematically

$$\text{Electric flux } (\Phi) = \frac{1}{\epsilon_0} q$$

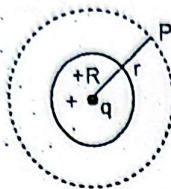
$$\oint \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} q$$

Gaussian surface is an imaginary surface which encloses the given charge. Gaussian surface is constructed to find the electric field due to a given charge at certain point.

Electric field due to non-conducting spherically symmetric charge distribution of radius R is calculated as follows:

i. **Outside the Sphere**

Let P be a point at distance r from the centre of charge distribution where electric field has to be determined. Now, construct a gaussian sphere of radius r through point P .



Applying Gauss law,

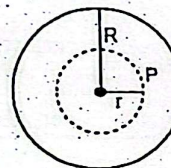
$$\oint \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0}$$

$$E(4\pi r^2) = \frac{q}{\epsilon_0}$$

$$\therefore E = \frac{q}{4\pi\epsilon_0 r^2}$$

ii. **Inside the sphere.**

Let P be the point at distance r from the centre of sphere. Construct gaussian sphere of radius r through point P .



Let q' be the charge enclosed by the gaussian surface.

Using Gauss law,

$$\oint \vec{E} \cdot d\vec{s} = \frac{q'}{\epsilon_0}$$

$$\text{or, } E 4\pi r^2 = \frac{q'}{\epsilon_0}$$

$$\therefore E = \frac{q'}{4\pi\epsilon_0 r^2}$$

Since volume charge density is same for two spheres, we can write

$$\frac{q}{\frac{4}{3}\pi R^3} = \frac{q'}{\frac{4}{3}\pi r^3}$$

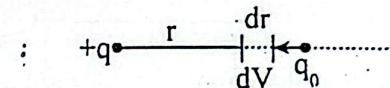
$$\text{or, } q' = \frac{r^3}{R^3} q$$

Electric field becomes

$$E = \frac{r^3}{R^3} \frac{q}{4\pi\epsilon_0 r^2}$$

$$\therefore E = \frac{qr}{4\pi\epsilon_0 R^3}$$

5.5 Relation Between Electric Field and Potential



Consider a test charge q_0 is moving along given charge q . On moving a small distance dr , small work done against the field is dw , and change in potential is dV .

Small work done against the field is $(dw) = -F dr$

$$\text{or, } q_0 dV = -E q_0 dr$$

$$\therefore E = -\frac{dV}{dr}$$

$$\text{In three dimension, } \vec{E} = -\left(\frac{\partial V}{\partial x} \vec{i} + \frac{\partial V}{\partial y} \vec{j} + \frac{\partial V}{\partial z} \vec{k}\right) \\ = -\text{gradient } V$$

$$\therefore \vec{E} = -\nabla V$$

Comparing $\vec{E} = -\left(\frac{\partial V}{\partial x} \vec{i} + \frac{\partial V}{\partial y} \vec{j} + \frac{\partial V}{\partial z} \vec{k}\right)$ with $\vec{E} = E_x \vec{i} + E_y \vec{j} + E_z \vec{k}$, we have

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad \text{and } E_z = -\frac{\partial V}{\partial z}$$

This is the relationship between electric field and electric potential.

5.6 Capacitor with Dielectric

Capacitance of a capacitor can be increased by inserting a dielectric medium in between the two plates of capacitor by a factor equal to the dielectric constant of the medium.

Capacitance with dielectric (C') = $k C$

Where k is a dielectric constant and C is a capacitance without dielectric.

Consider a dielectric slab is inserted between the plates of a parallel plate capacitor having charges $+q$ and $-q$. Induced charges $-q'$ and $+q'$ are observed at lower surface and upper surface of the dielectric. To find the electric field between the plates of capacitor let us construct a Gaussian surface enclosing the upper plate of capacitor as shown in figure.



Now, applying Gauss's law

$$\oint \vec{E} \cdot d\vec{S} = \frac{q - q'}{\epsilon_0}$$

$$\text{or, } E A = \frac{q - q'}{\epsilon_0}$$

$$\text{or, } \epsilon_0 E = \frac{q - q'}{A} = \frac{q}{A} - \frac{q'}{A}$$

$$\therefore \epsilon_0 E = D - P$$

In vector form,

$$\epsilon_0 \vec{E} = \vec{D} - \vec{P}$$

where $\vec{E}$ is an electric field vector, $\vec{D}$ is a displacement vector and is equivalent to surface charge density, and $\vec{P}$ is a polarization vector and is equivalent to induced surface charge density.

5.7 Dielectrics

Dielectrics are the materials that conduct electrons or charges but does not conducts electricity. On the basis of behavior shown by dielectric placed in an electric field there are two types of dielectric:

- **Polar dielectric**

The dielectrics (molecules) in which the centre of gravity of positive charge does not coincides with that of negative charges are called *polar molecules*. Such molecules have permanent dipole moment. Familiar examples of polar molecules are H_2O , NH_3 , HCL etc.

- **Non-polar dielectric**

The dielectrics (molecules) in which the centre of gravity of positive charge coincides with that of negative charges are called *non-polar molecules*. Such molecules do not have permanent dipole moment. Familiar examples of polar molecules are H_2 , N_2 , CO_2 , CH_4 etc.

Polarization

When a dielectric material is inserted between the plates of capacitor some charges induced on dielectric slab resulting polarization which can be defined as the induced charge per unit area or induced dipole moment per unit volume and given by

$$P = \frac{q'}{A}$$

$$= \frac{q'l}{Al} = \frac{\text{Dipole moment}}{\text{Volume}}$$

In terms of permittivity and susceptibility,

$$P = \epsilon_0 \chi_e E$$

Where ϵ_0 is permittivity of free space and χ_e is electric susceptibility.

Polarizability

Atomic polarizability is defined as the electric dipole moment induced in the atom by an electric field of unit strength. The electronic contribution to the polarizability is due to the displacement of electrons in an atom relative to the nucleus.

5.8 Clausius-Mossotti Equation

When an electric field E is applied to a material, displacement of positive and negative charge takes place and thus electric dipole moment is created. This phenomenon is called polarization. Polarization is defined as the total induced dipole moment per unit volume. For an ordinary electric field, the polarizability is given by

$$P = \epsilon_0 \chi_e E \dots\dots\dots(i)$$

Where ϵ_0 is permittivity of free space and χ_e is electric susceptibility.

$$\text{Also } P = N\alpha\epsilon_0 E = N\alpha E_{\text{eff}} \dots\dots\dots(ii)$$

Where N is number of atoms per unit volume, α is atomic polarizability and E_{eff} is effective field. The value of effective field for dielectric medium is given by

$$E_{\text{eff}} = E + \frac{P}{3\epsilon_0} \dots\dots\dots(iii)$$

And the displacement vector is $D = \epsilon_0 E + P$

$$\text{or, } \epsilon_r \epsilon_0 E = \epsilon_0 E + P$$

$$\therefore P = \epsilon_0 E(\epsilon_r - 1)$$

Where ϵ_r is dielectric constant.

Now, equation (ii) becomes,

$$E_{\text{eff}} = \frac{P}{N\alpha}$$

$$= \frac{\epsilon_0 E(\epsilon_r - 1)}{N\alpha} \dots\dots\dots(iv)$$

From equation (iii),

$$E_{\text{eff}} = E + \frac{\epsilon_0 E(\epsilon_r - 1)}{3\epsilon_0}$$

$$= \frac{E(\epsilon_r + 2)}{3} \dots\dots\dots(v)$$

Comparing (iv) and (v)

$$\frac{\epsilon_0 E(\epsilon_r - 1)}{N\alpha} = \frac{E(\epsilon_r + 2)}{3}$$

$$\text{or, } \frac{3\epsilon_0}{N\alpha} = \frac{(\epsilon_r + 2)}{(\epsilon_r - 1)}$$

$$\text{Hence, } \frac{N\alpha}{3\epsilon_0} = \frac{(\epsilon_r - 1)}{(\epsilon_r + 2)} \dots\dots\dots(vi)$$

This is the Clausius-Mossotti equation giving the relation between polarizability and dielectric constant of solids.

SOLUTION TO EXAM PROBLEMS AND IMPORTANT QUESTIONS

1. Charges of uniform volume density $3.2 \mu\text{C}/\text{m}^3$ fill a non-conducting solid sphere of radius 5 cm. What is the magnitude of the electric field at (a) 3.5 cm (b) 8 cm from the centre of the sphere?

⇒ Volume charge density (ρ) = $3.2 \mu\text{C}/\text{m}^3$

Radius of sphere (R) = 5 cm = 0.05 m

$$\begin{aligned}\text{Charge in sphere (Q)} &= \frac{4}{3} \pi R^3 \rho \\ &= \frac{4}{3} \pi (5 \times 10^{-2})^3 \times 3.2 \times 10^{-6} \\ &= 1.67 \times 10^{-9} \text{ C}\end{aligned}$$

$$\text{Electric field, } E = \frac{Q}{4\pi\epsilon_0 r^2}$$

At $r = 3.5$ cm,

$$\begin{aligned}E &= \frac{1.67 \times 10^{-9}}{4\pi \times 8.85 \times 10^{-12} \times (3.5 \times 10^{-2})^2} \\ &= 1.22 \times 10^4 \text{ N/C}\end{aligned}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

At $r = 8$ cm,

$$\begin{aligned}E &= \frac{1.67 \times 10^{-9}}{4\pi \times 8.85 \times 10^{-12} \times (8 \times 10^{-2})^2} \text{ N/C} \\ &= 2.34 \times 10^3 \text{ N/C}\end{aligned}$$

2. A neutral water molecule in its vapour state has an electric dipole moment of magnitude 7.1×10^{-30} Cm. If the molecule is placed in an electric field of 2.5×10^4 N/C, (i) What maximum torque can the field exert on it? (ii) How much work must an external agent do to turn this molecule end for end in this field?

⇒ Dipole moment (P) = 7.1×10^{-30} Cm

Electric field (E) = 2.5×10^4 N/C

(i) Maximum torque is given by

$$\begin{aligned}\tau &= PE \sin 90^\circ \\ &= 7.1 \times 10^{-30} \times 2.5 \times 10^4 \\ &= 17.75 \times 10^{-26} \text{ Nm}\end{aligned}$$

(ii) The work done by torque dw during infinitesimal displacement $d\theta$ is

$$dw = \tau d\theta$$

$$\text{Total torque done is } w = \int_{\theta_1}^{\theta_2} dw$$

$$= \int_0^{180^\circ} PE \sin \theta d\theta$$

$$= -PE [\cos \theta]_0^{180^\circ}$$

$$= -PE [\cos 180^\circ - \cos 0^\circ]$$

$$= 2PE$$

$$= 2 \times 17.75 \times 10^{-26}$$

$$= 35.5 \times 10^{-26} \text{ Joule}$$

3. What is the magnitude of the electric field at the point (2, 3) m if the electric potential is given by $V = 2x + 5xy + 3y^2$ volts? What acceleration does an electron experience in the x-direction?

$$\Rightarrow V(x, y) = 2x + 5xy + 3y^2$$

$$\text{We have, } E_x = -\frac{\partial V}{\partial x}$$

$$= -(2 + 5y)$$

Electric field at (2, 3) is given as

$$E_x(2, 3) = -(2 + 5 \times 3) = -17 \text{ V/m}$$

Force experienced by electron is $(F) = -e E_x$

$$\text{or, } ma = 17 \times 1.6 \times 10^{-19}$$

$$\therefore a = \frac{17 \times 1.6 \times 10^{-19}}{9.1 \times 10^{-31}}$$

$$= 2.98 \times 10^{12} \text{ m/s}^2$$

4. A spherical drop of water carrying a charge of 30 pC has a potential of 500 V at its surface (with $V = 0$ at infinity). (a) What is the radius of the drop? (b) If two such drops of the same charge and radius combine to form a single spherical drop, what is the potential at the surface of the new drop?

$$\Rightarrow \text{a. Potential } (V) = \frac{q}{4\pi\epsilon_0 R}$$

$$\text{or, } R = \frac{q}{4\pi\epsilon_0 V}$$

$$= \frac{30 \times 10^{-12}}{4\pi \times 8.85 \times 10^{-12} \times 500} = 0.54 \text{ mm}$$

- b. If V be the potential at the surface of small drop, then

$$V = \frac{q}{4\pi\epsilon_0 R} \quad \dots\dots\dots (1)$$

If V' be the potential at the surface of new drop, then

$$V' = \frac{Q}{4\pi\epsilon_0 R'} \quad \dots\dots\dots (2)$$

Dividing equation (2) by (1), we get

$$\frac{V'}{V} = \frac{Q}{R'} \times \frac{R}{q}$$

$$= \frac{2q}{q} \times \frac{R}{R'} = \frac{2R}{R'} \quad \dots\dots\dots (3)$$

Since the volume of new drop equals the volume of small drop, we can write

$$\frac{4}{3} \pi R^3 \times 2 = \frac{4}{3} \pi R'^3$$

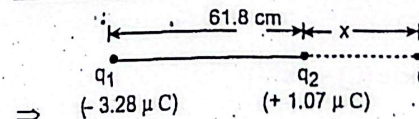
$$\therefore R' = 2^{1/3} R$$

From equation (3), we get

$$V' = 2^{2/3} V$$

$$= 2^{2/3} \times 500 = 793.7 \text{ V}$$

5. Two fixed charges $1.07 \mu\text{C}$ and $-3.28 \mu\text{C}$ are 61.8 cm apart, where may a third charge be located so that no net force acts on it?



Let a third charge q be located at a distance x from q_2 .

Force on q due to q_1 is

$$F_1 = \frac{qq_1}{4\pi\epsilon_0 (0.618 + x)^2} \quad \dots\dots\dots (1)$$

Force on q due to q_2 is

$$F_2 = \frac{qq_2}{4\pi\epsilon_0 x^2} \quad \dots\dots\dots (2)$$

Net force on q is

According to the question, $F = 0$. So,

$$F_2 = F_1$$

$$\text{or, } \frac{qq_2}{4\pi\epsilon_0 x^2} = \frac{qq_1}{4\pi\epsilon_0 (0.618 + x)^2}$$

$$\text{or, } \frac{1.07 \times 10^{-6}}{x^2} = \frac{3.28 \times 10^{-6}}{(0.618 + x)^2}$$

$$\therefore x = 0.823 \text{ m}$$

6. Two small, positively charged spheres have a combined charge of 5.0×10^{-5} C. If each sphere is repelled from other by an electrostatic force of 1.0 N when the spheres are 2.0 m apart, what is the charge on the sphere with the smaller charge?

⇒ Combined charge on sphere, $q_1 + q_2 = 5 \times 10^{-5}$ C

Force of repulsion (F) = 1 N

Separation (d) = 2 m

Charge on the sphere with smaller charge = ?

Given, $q_1 + q_2 = 5 \times 10^{-5}$ (1)

We have,

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 d^2}$$

$$\text{or, } q_1 q_2 = 4\pi\epsilon_0 d^2 \times F = \frac{1}{9 \times 10^9} \times 4 \times 1$$

$$= 0.44 \times 10^{-9} \text{ (2)}$$

$$\begin{aligned} q_1 - q_2 &= \sqrt{(q_1 + q_2)^2 - 4q_1 q_2} \\ &= \sqrt{(5 \times 10^{-5})^2 - 4 \times 0.44 \times 10^{-9}} \\ &= 0.72 \times 10^{-5} \text{ C (3)} \end{aligned}$$

Solving equations (1) and (2)

$$2q_1 = 7.72 \times 10^{-5}$$

$$\Rightarrow q_1 = 3.86 \times 10^{-5} \text{ C}$$

Charge on sphere $q_2 = q - q_1$

$$= 5 \times 10^{-5} - 3.86 \times 10^{-5}$$

$$= 1.14 \times 10^{-5} \text{ C}$$

7. Charge of uniform volume density $\rho = 3.2 \mu\text{C/m}^3$ fills a non-conducting solid sphere of radius 5 cm. What is the magnitude of the electric field 3 cm from the sphere's center?

⇒ Since the charge distribution is uniform, the total charge q is obtained as

$$q = \rho \times \frac{4}{3} \pi R^3$$

$$= \frac{4}{3} \times 3.2 \times 10^{-6} \times \pi \times (5 \times 10^{-2})^3$$

$$= 1.6755 \times 10^{-9} \text{ C}$$

$$\text{Now, } E_{\text{inside}} = \frac{|q| r}{4\pi\epsilon_0 R^3}$$

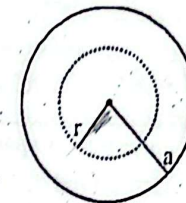
$$= \frac{1.6755 \times 10^{-9} \times 3 \times 10^{-2}}{4\pi \times 8.854 \times 10^{-12} \times (5 \times 10^{-2})^3}$$

$$= 3614.15 \text{ V/m}$$

8. A spherical charge distribution has volume charge density (a) $\rho = Ar^n$ at $r < a$ (b) $\rho = \rho_0$ for $r > a$, where a is the radius of the sphere and the electric field in both cases.

⇒ Here charge density $\rho = Ar^n$ $r < a$
 $= \rho_0$ $r > a$

i. From Gauss' law, we have



$$\oint \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0}$$

$$\text{or, } E(4\pi r^2) = \frac{1}{\epsilon_0} \int_0^r \rho dv$$

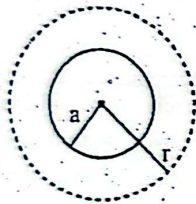
$$\text{or, } E(4\pi r^2) = \frac{1}{\epsilon_0} \int_0^r (Ar^n) 4\pi r^2 dr$$

$$\text{or, } E = \frac{1}{\epsilon_0 r^2} \int_0^r Ar^{n+2} dr = \frac{1}{\epsilon_0 r^2} \frac{Ar^{n+3}}{(n+3)}$$

$$\therefore E = \frac{Ar^{n+1}}{\epsilon_0 (n+3)}$$

$$\text{ii. } \oint \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0}$$

$$\text{or, } E(4\pi r^2) = \frac{1}{\epsilon_0} \int_0^a \rho dv$$



$$\text{or, } E = \frac{1}{4\pi\epsilon_0 r^2} \int_0^a \rho_0 4\pi r^2 dr$$

$$\text{or, } E = \frac{\rho_0 4\pi a^3}{4\pi 3\epsilon_0 r^2}$$

$$\therefore E = \frac{q}{4\pi\epsilon_0 r^2} \quad \left[\because q = \rho_0 V = \frac{4}{3}\pi a^3 \rho_0 \right]$$

9. Over certain region of space the electric potential is $V = 15x - 3x^2y + 12yz^2$. Find the expression for the x, y and z components of the electric field over this region. What is the magnitude of the field at the point P that has coordinates (1, 0, -2) m?

⇒ Electric field along x-axis is

$$E_x = -\frac{\partial V}{\partial x}$$

$$= -(15 - 6xy) = 6xy - 15$$

Electric field along y-axis is

$$E_y = -\frac{\partial V}{\partial y}$$

$$= -(-3x^2 + 12z^2) = 3x^2 - 12z^2$$

Electric field along z-axis is

$$E_z = -\frac{\partial V}{\partial z}$$

$$= -24yz$$

At point (1, 0, -2),

$$E_x = -15 \text{ V/m}, E_y = -45 \text{ V/m}, E_z = 0$$

Hence, the magnitude of electric field is

$$E = \sqrt{15^2 + 45^2} \\ = 47.43 \text{ V/m}$$

10. Two large parallel plates are separated by a distance of 5 cm. The plates have equal but opposite charges that create an electric field in the region between the plates. An alpha particle ($q = 3.2 \times 10^{-19} \text{ C}$, $m = 6.68 \times 10^{-27} \text{ kg}$) is released from the positively charged plate, and it strikes the negatively charged plate $2 \times 10^{-6} \text{ sec}$ later. Assuming that the electric field between plates is uniform and perpendicular to the plates, what is the strength of electric field?

⇒ Here, separation between the plates (d) = 5 cm = 0.05 m
 Mass of an alpha particle (m) = $6.68 \times 10^{-27} \text{ kg}$
 Charge on alpha particle (q) = $3.2 \times 10^{-19} \text{ C}$
 Time taken (t) = $2 \times 10^{-6} \text{ sec}$
 Electric field strength (E) = ?

We know force exerted on particle $F = Eq = ma$

$$\text{Acceleration (a)} = \frac{Eq}{m}$$

$$\text{And distance travel } s = ut + \frac{1}{2}at^2 \quad (\because u = 0)$$

$$\text{or, } s = \frac{1}{2} \left(\frac{Eq}{m} \right) t^2$$

$$E = \frac{2ms}{qt^2}$$

$$= \frac{2 \times 6.68 \times 10^{-27} \times 0.05}{3.2 \times 10^{-19} \times (2 \times 10^{-6})^2}$$

$$E = 521.8 \text{ Vm}^{-1}$$

11. What is the magnitude of a point charge chosen so that the electric field 50 cm away has the magnitude of 2 N/C.

[2013 Fall]

$$\Rightarrow \text{We know, electric field of point charge is, } E = \frac{q}{4\pi\epsilon_0 r^2}$$

$$\text{or, } q = E \times 4\pi\epsilon_0 r^2$$

$$= 2 \times 4\pi \times 8.85 \times 10^{-12} \times 0.5^2$$

$$\therefore q = 55.6 \text{ pC}$$

12. Find the electric field intensity at the centre of a square of side 5 cm consisting of $2 \mu\text{C}$ charges in each vertex.

[2016 Spring]

$$\Rightarrow \text{The distance between centre and corners of square is } r$$

$$= \frac{a}{\sqrt{2}}$$

$$= \frac{5}{\sqrt{2}}$$

$$= 3.53 \text{ cm} = 0.0353 \text{ m}$$

Electric field produced by one of the charge is E

$$= \frac{q}{4\pi\epsilon_0 r^2}$$

$$= \frac{9 \times 10^9 \times 2 \times 10^{-6}}{0.0353^2}$$

$$= 14.4 \times 10^6 \text{ N/C}$$

Since all the charges at four corners are equal resulting a net electric field zero.

13. The magnitude of the average electric field normally present in the Earth's atmosphere just above the surface of the earth is about 150 N/C, directed downward (radially inward, towards the centre of earth). What is the total net surface charge carried by the Earth? Assume the earth to be a conductor. [2018 Fall]

$$\Rightarrow \text{We know, electric field due to a charged sphere is}$$

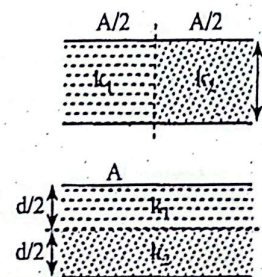
$$E = \frac{q}{4\pi\epsilon_0 r^2}$$

$$\text{or, } q = E \times 4\pi\epsilon_0 r^2$$

$$= 150 \times 4\pi \times 8.85 \times 10^{-12} \times (6.4 \times 10^6)^2$$

$$\therefore q = 6.8 \times 10^3 \text{ C}$$

14. A parallel plate capacitor contains two dielectric slabs (of equal dimensions) of dielectrics k_1 and k_2 as shown in figure below. (i) Find the capacitance in each case if A is the area of each plate. (ii) If $k_1 = 2$ and $k_2 = 3$, what will be the ratio of the capacitance in two cases?



$\Rightarrow$

i. Case I:

$$C_1 = \frac{k_1 \epsilon_0 \frac{A}{2}}{d}$$

$$= \frac{k_1 \epsilon_0 A}{2d}$$

$$C_2 = \frac{k_2 \epsilon_0 \frac{A}{2}}{d}$$

$$= \frac{k_2 \epsilon_0 A}{2d}$$

Equivalent capacitance due to parallel combination of capacitors can be calculated as

$$C = C_1 + C_2$$

$$= \frac{\epsilon_0 A}{d} \left(\frac{k_1 + k_2}{2} \right)$$

Case II:

$$C_1 = \frac{k_1 \epsilon_0 A}{\frac{d}{2}}, C_2 = \frac{k_2 \epsilon_0 A}{\frac{d}{2}}$$

Equivalent capacitance due to series combination of capacitors can be calculated as

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\text{or, } C = \frac{C_1 C_2}{C_1 + C_2}$$

$$\text{or, } C = \frac{\frac{k_1 \epsilon_0 A}{2d} \cdot \frac{k_2 \epsilon_0 A}{2d}}{\frac{k_1 \epsilon_0 A}{2d} + \frac{k_2 \epsilon_0 A}{2d}}$$

$$\therefore C = \frac{2\epsilon_0 A}{d} \left(\frac{k_1 k_2}{k_1 + k_2} \right)$$

$$\begin{aligned} \text{ii. Ratio of capacitance} &= \frac{\frac{\epsilon_0 A}{d} \left(\frac{2}{2} \right)}{\frac{2\epsilon_0 A}{d} \left(\frac{k_1 k_2}{k_1 + k_2} \right)} \\ &= \frac{25}{24} = 25:24 \end{aligned}$$

15. If a parallel plate capacitor is to be designed to operate in an environment of fluctuating temperature, prove that the rate of change of capacitance with temperature T is given by: $\frac{dC}{dT} = C \left[\frac{1}{A} \frac{dA}{dT} - \frac{1}{x} \frac{dx}{dT} \right]$ where symbols carry their usual meanings.

$\Rightarrow$ Capacitance of a parallel plate capacitor is

$$C = \epsilon_0 \frac{A}{x} \dots (i)$$

where A is area of plate and x is separation of plate.

Differentiating w.r.t. temperature, we get

$$\frac{dC}{dT} = \epsilon_0 \frac{d}{dT} \left(\frac{A}{x} \right)$$

$$= \epsilon_0 \left[\frac{\frac{dA}{dT} x - A \frac{dx}{dT}}{x^2} \right]$$

$$= \epsilon_0 \left[\frac{1}{x} \frac{dA}{dT} - \frac{A}{x^2} \frac{dx}{dT} \right]$$

$$= \epsilon_0 \frac{A}{x} \left[\frac{1}{A} \frac{dA}{dT} - \frac{1}{x} \frac{dx}{dT} \right]$$

$$\therefore \frac{dC}{dT} = C \left[\frac{1}{A} \frac{dA}{dT} - \frac{1}{x} \frac{dx}{dT} \right] \text{ proved.}$$

16. A cylindrical capacitor has radii a and b . Show that half the stored electric potential energy lies within a cylinder of radius $r = \sqrt{ab}$.

⇒ Capacitance of capacitor having internal radius a and external radius b is

$$C = \frac{2\pi\epsilon_0 l}{\ln(b/a)} \quad \dots\dots\dots(i)$$

Capacitance of capacitor with inner radius a and outer radius $\sqrt{ab}$ is

$$\begin{aligned} C' &= \frac{2\pi\epsilon_0 l}{\ln(\sqrt{ab}/a)} \\ &= \frac{2\pi\epsilon_0 l}{\ln(\sqrt{b/a})} \\ &= \frac{4\pi\epsilon_0 l}{\ln(b/a)} \quad \dots\dots\dots(ii) \end{aligned}$$

Now, energy in first case is

$$\begin{aligned} E &= \frac{q^2}{2C} \\ &= \frac{q^2}{2 \times 2\pi\epsilon_0 l} \ln(b/a) \quad \dots\dots\dots(iii) \end{aligned}$$

Energy in second case,

$$E' = \frac{q^2}{2 \times 4\pi\epsilon_0 l} \ln(b/a) \quad \dots\dots\dots(iv)$$

Dividing (iv) by (iii), we get

$$\frac{E'}{E} = \frac{1}{2}$$

$$\therefore E' = \frac{1}{2} E$$

Hence, half the stored electric potential energy lies within a cylinder of radius $r = \sqrt{ab}$.

17. A capacitor of capacitance ' C ' is discharged through a resistor of resistance ' R '. After how many time constants is the energy stored becomes one fourth of initial value?

⇒ Energy stored in the capacitor at any time is given by

$$\begin{aligned} U &= \frac{q^2}{2C} \\ &= \frac{q_0^2}{2C} e^{\frac{-2t}{RC}} = U_0 e^{\frac{-2t}{RC}} \quad [\because q = q_0 e^{\frac{-t}{RC}}] \end{aligned}$$

where $U_0 = \frac{1}{2C} q_0^2$ is initial energy stored in the capacitor.

$$\text{We have, } U = \frac{U_0}{4}$$

$$\text{or, } \frac{U_0}{4} = U_0 e^{\frac{-2t}{RC}}$$

$$\text{or, } \ln(4) = \frac{2t}{RC}$$

$$t = \frac{RC \ln(4)}{2}$$

$$= \frac{\tau_c \ln 4}{2}; \quad \tau_c = RC \text{ is capacitive time constant.}$$

$$\therefore t = 0.693 \tau_c$$

18. A long cylindrical conductor has length l m and is surrounded by a co-axial cylindrical conducting shell with inner radius double that of long cylindrical conductor. Calculate the capacitance for this capacitor assuming that there is vacuum in space between cylinders.

⇒ Let $a = x$ and $b = 2x$ be the inner and outer radii of cylindrical conducting shell.

$$\text{Capacitance, } C = \frac{2\pi\epsilon_0 l}{\ln(b/a)}$$

$$= \frac{2\pi\epsilon_0 \times 1}{\ln 2}$$

$$= \frac{2\pi \times 8.85 \times 10^{-12}}{\ln 2}$$

$$= 80.22 \text{ pF}$$

19. The space between two concentric conducting spherical shells of radii $b = 1.70 \text{ cm}$ and $a = 1.20 \text{ cm}$ is filled with a substance of dielectric constant $k = 23.5$. A potential difference $V = 73 \text{ V}$ is applied across the inner and outer shells. Determine (a) the capacitance of the device (b) the free charge q on the inner shell.

⇒ Radius of inner shell (a) = $1.2 \text{ cm} = 0.012 \text{ m}$
 Radius of outer shell (b) = $1.7 \text{ cm} = 0.017 \text{ m}$
 Dielectric constant (k) = 23.5
 Potential difference (V) = 73 V
 Capacitance of device (C) = ?
 Free charge (q) = ?
 Capacitance of spherical capacitor,

$$C = \frac{k4\pi\epsilon_0 ab}{b-a}$$

$$= 23.5 \times \frac{1}{9 \times 10^9} \times \frac{0.012 \times 0.017}{0.017 - 0.012}$$

$$= 1.06 \times 10^{-10} \text{ F}$$

$$\text{Free charge (q)} = CV$$

$$= 1.06 \times 10^{-10} \times 73$$

$$= 7.78 \times 10^{-9} \text{ C}$$

20. A parallel plate capacitor has a capacitance of $100 \mu\text{F}$, a plate area of 100 cm^2 and a mica dielectric. At 50 volts potential difference, calculate (a) E in the mica (b) the free charge on the plates (c) the induced surface charge. [Dielectric constant for mica, $k = 5.4$]

$$\Rightarrow \text{a. } E = \frac{q}{k\epsilon_0 A}$$

$$= \frac{5 \times 10^{-3}}{5.4 \times 8.854 \times 10^{-12} \times 100 \times 10^{-4}}$$

$$= 0.1046 \times 10^{11} \text{ V/m}$$

$$\text{b. } q = CV$$

$$= 100 \times 10^{-6} \times 50$$

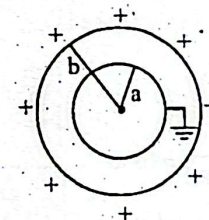
$$= 5 \times 10^{-3} \text{ C}$$

$$\text{c. } q' = q \left(1 - \frac{1}{k}\right)$$

$$= 5 \times 10^{-3} \left(1 - \frac{1}{5.4}\right)$$

$$= 4.074 \times 10^{-3} \text{ C}$$

21. Prove the capacitance of a concentric spherical capacitor of radii a and b is $C = 4\pi\epsilon_0 \left(\frac{b^2}{b-a}\right)$ if outer plate is positively charged and inner sphere is earthed.



⇒ The system can be assumed to be an isolated single sphere of capacitance $C_1 = 4\pi\epsilon_0 b$(i)

And a concentric capacitor of radii a and b having capacitance

$$C_2 = 4\pi\epsilon_0 \left(\frac{ab}{b-a}\right) \text{(ii)}$$

Two capacitors are in parallel combination with equivalent capacitance $C = C_1 + C_2$

$$C = 4\pi\epsilon_0 b + 4\pi\epsilon_0 \left(\frac{ab}{b-a} \right)$$

$$= \frac{4\pi\epsilon_0 b^2 - 4\pi\epsilon_0 ab + 4\pi\epsilon_0 ab}{b-a}$$

$$\text{Hence, } C = 4\pi\epsilon_0 \left(\frac{b^2}{b-a} \right)$$

22. Obtain the charging time constant of a capacitor in a RC circuit such that current through the resistor is decreased by 50% of its peak value in 5 seconds.

[2014 Fall, 2019 Spring]

$$\Rightarrow \text{Current through the circuit is } I = I_0 e^{-t/RC}$$

$$\text{or, } 0.5 I_0 = I_0 e^{-t/RC}$$

$$\text{or, } e^{t/RC} = 2$$

$$\text{or, } \frac{t}{RC} = \ln 2$$

$$\therefore \text{Charging time constant is } (RC) = 7.21 \text{ s}$$

23. If the charge on a capacitor is increased by 2 coulomb, the energy stored in it is increased by 21%. Find the original charge on the capacitor.

[2016 Fall]

$$\Rightarrow \text{Let } q \text{ be the initial charge stored in capacitor the initial energy stored in capacitor is } E = \frac{q^2}{2C} \dots\dots\dots (i)$$

$$\text{And final energy stored is } E' = \frac{(q+2)^2}{2C} \dots\dots\dots (ii)$$

$$\text{From eq (i) and (ii) } E' = 1.21 E$$

$$\text{or, } (q+2)^2 = 1.21 q^2$$

$$\therefore q+2 = 1.1q$$

$$\therefore \text{Initial charge in capacitor is } q = 20 \text{ C}$$

24. A parallel plate capacitor has a capacitance of $100 \times 10^{-12} \text{ F}$, a plate of area of 100 cm^2 mica ($k=7$) is used as a dielectric, at 50 V potential difference. Calculate the electric field intensity and magnitude of induced charge.

[2018 Spring]

$$\Rightarrow \text{Free charge stored on capacitor is } q = CV = 100 \times 10^{-12} \text{ F} \times 50 \text{ V}$$

$$= 5 \times 10^{-9} \text{ C}$$

$$\text{Capacitance of parallel plate capacitor } C = \frac{\epsilon_0 A}{d}$$

$$\text{or, plate separation } (d) = \frac{\epsilon_0 A}{C}$$

$$\text{Electric field } (E) = V/d$$

$$= \frac{VC}{\epsilon_0 A}$$

$$= \frac{50 \times 100 \times 10^{-12}}{8.85 \times 10^{-12} \times 100 \times 10^{-4}}$$

$$= 56497.17 \text{ V/m}$$

$$\text{Induced charge on capacitor } (q') = \left(\frac{k-1}{k} \right) q$$

$$= 4.28 \text{ nC}$$

25. A parallel plate capacitor has circular plates of 8 cm radius and 1 mm separation. What charge will appear on the plates if a p. d. of 100 V is applied

[2017 Fall]

$$\Rightarrow \text{Charge stored on capacitor is } q = CV$$

$$\text{or, } q = \frac{\epsilon_0 A}{d} \times V \quad [\therefore C = \frac{\epsilon_0 A}{d} \text{ for parallel plate capacitor}]$$

$$= \frac{8.85 \times 10^{-12} \times \pi \times 0.08^2}{0.001} \times 100$$

$$\therefore q = 17.8 \text{ nC}$$

26. Two 10 Coulombs and 40 Coulombs are 12 cm apart. Find the position of point where intensity is zero.

[2021 Fall]

⇒ Let x be the point from 10 C charge where the intensity is zero.

We know the electric field intensity given by charge q_1 is

$$E_1 = \frac{q_1}{4\pi\epsilon_0 x^2} \dots\dots\dots (1)$$

Similarly electric field intensity given by charge q_2 is

$$E_2 = \frac{q_2}{4\pi\epsilon_0 (12-x)^2} \dots\dots\dots (2)$$

According to the question, $E_1 = E_2$

$$\text{or, } \frac{q_1}{4\pi\epsilon_0 x^2} = \frac{q_2}{4\pi\epsilon_0 (12-x)^2}$$

$$\text{or, } \frac{10}{x^2} = \frac{40}{(12-x)^2}$$

$$\text{or, } 4x^2 = (12-x)^2$$

$$\text{or, } x^2 + 8x - 48 = 0$$

$$\text{or, } (x+12)(x-4) = 0$$

$$\therefore x = 4, -12 \text{ cm}$$

Hence, intensity is zero at a distance of 4 cm right from 10 C and at a distance 12 cm left from 10 C.

27. A cylindrical resistor of radius 6 mm and length 2.5 cm is made of material that has a resistivity of $4 \times 10^{-5} \Omega \text{ m}$. What are (i) the magnitude of the current density and (ii) the potential difference when the energy dissipation rate in the resistor is 2 watt?

$$\Rightarrow \text{Radius of resistor (r)} = 6 \text{ mm} = 6 \times 10^{-3} \text{ m}$$

$$\text{length (l)} = 2.5 \text{ cm} = 2.5 \times 10^{-2} \text{ m}$$

$$\text{Resistivity } (\rho) = 4 \times 10^{-5} \Omega \text{ m}$$

$$\text{i. We have } P = I^2 R = J^2 A^2 R$$

$$\text{or, } J = \frac{1}{A} \sqrt{\frac{P}{R}}$$

$$\text{or, } J = \frac{1}{A} \sqrt{\frac{P A}{\rho l}} \quad \left[\because R = \frac{\rho l}{A} \right]$$

$$\text{or, } J = \sqrt{\frac{P}{\rho l A}}$$

$$= \sqrt{\frac{2}{4 \times 10^{-5} \times 2.5 \times 10^{-2} \times \pi (6 \times 10^{-3})^2}}$$

$$= 1.33 \times 10^5 \text{ A/m}^2$$

$$P = VI = JAV$$

$$\therefore V = \frac{P}{JA}$$

$$= \frac{2}{\pi (6 \times 10^{-3})^2 \times 1.33 \times 10^5} = 0.133 \text{ V}$$

28. Calculate the drift speed of electrons when 20 A current is supplied through a copper wire of cross-sectional area 1 mm^2 and electron density 10^{28} m^{-3} .

⇒ Drift speed (v) = ?

$$\text{Current (I)} = 20 \text{ A}$$

$$\text{Cross sectional area (A)} = 1 \text{ mm}^2$$

$$= 1 \times \text{mm}^2$$

$$= 1 \times 10^{-6} \text{ m}^2$$

$$\text{Electron density of copper (n)} = 10^{28} \text{ m}^{-3}$$

$$\text{Using, } I = venA$$

$$= \frac{I}{neA}$$

$$= \frac{20}{10^{28} \times 10^{-19} \times 10^{-6}}$$

$$= 0.0125 \text{ A} = 12.5 \text{ mA}$$

29. A wire of resistance of 6Ω is stretched by 20% what is the resistance of wire? Assume that resistivity remains same [2018 Spring]

$\Rightarrow$ Resistance of conductor is $R = \frac{\rho L}{A}$ (i)

or, $6 = \frac{\rho L}{A}$

$\therefore$ Resistance of longer wire, $R' = \frac{\rho(1.2L)}{A}$
 $= 1.2 \times 6 = 7.2\Omega$

30. What are the
 i. Mean time between collisions?
 ii. The mean free path for free electrons in copper?
 Given $n = 8.4 \times 10^{28} \text{ m}^{-3}$, $\rho = 1.7 \times 10^{-8} \Omega\text{m}$, $v_{\text{avg}} = 1.6 \times 10^6 \text{ m/s}$. [2017 Fall]

$\Rightarrow$ Resistivity of conductor is $\rho = \frac{m}{ne^2\tau}$

i. mean time (τ) = $\frac{m}{ne^2\rho}$
 $= \frac{9.1 \times 10^{-31}}{8.4 \times 10^{28} \times (1.6 \times 10^{-19})^2 \times 1.7 \times 10^{-8}}$

$\therefore \tau = 2.49 \times 10^{-14} \text{ sec.}$

ii. Mean free path (λ) = $v_{\text{avg}} \times \tau$
 $= 1.6 \times 10^6 \times 2.49 \times 10^{-14}$
 $= 3.98 \times 10^{-8} \text{ m}$

31. There are two capacitors of capacitance $5\mu\text{F}$ and $2\mu\text{F}$. These capacitors are connect in series. The combination is connected to an inductor of inductance of 10mH . Calculate the resonance frequency of such combination. [2022 Fall]

$\Rightarrow$ Equivalent capacitance in series (C) = $\frac{C_1 C_2}{C_1 + C_2}$
 $= \frac{5 \times 2}{5 + 2}$
 $= \frac{10}{7} \mu\text{F} = 1.43 \mu\text{F}$

Now, the resonance frequency $f = \frac{1}{2\pi\sqrt{LC}}$
 $= \frac{1}{2\pi\sqrt{10 \times 10^{-3} \times 1.43 \times 10^{-6}}}$
 $= 1330.92 \text{ Hz}$

EXERCISE

1. Charge is distributed uniformly throughout the volume of an infinitely long cylinder of radius a . show that electric field at a distance r from the cylinder axis $r < a$ is given by $E = \frac{\rho r}{2\epsilon_0}$ where ρ is volume charge density.
2. Find the electric intensity outside and inside an infinitely long cylindrical charge distribution of radius a and uniform charge density ρ .

$$\text{Ans: } E_0 = \frac{\rho a^2}{2\epsilon_0 r}, \quad E_i = \frac{\rho r}{2\epsilon_0}$$
3. If a copper coin has mass 3.11 gm, what is the total charge on the nucleus of the atom in the coin? Also find the number of protons in the nucleus. Molar mass. (M) = 63.5 gm/mole, Avogadro number (N_A) = 6.02×10^{23} atom/mole.

$$\text{Ans: } 8.55 \times 10^{23}, 13.68 \times 10^4 C$$
4. Charge of uniform volume density $\rho = 3.2 \mu C/m^3$ fills on a non conducting solid sphere of radius 5cm. What is the magnitude of the electric field (i) at 3cm (ii) 8cm from the sphere's centre?

$$\text{Ans: } 3619.08 V/m, 188.5 V/m$$
5. Calculate the potential at (i) centre and (ii) $r = \frac{a}{2}$ for a spherical charge distribution $\rho = \rho_0$ for $r \leq a$ and $\rho = 0$ for $r > a$.

$$\text{Ans: } \frac{\rho_0 a^2}{2\epsilon_0}, \frac{2\rho_0 a^2}{15\epsilon_0}$$
6. 10.A capacitor C is discharged through resistor R . (a) After how many time constants does its charge fall to one half of its initial value? (b) After how many time constants does the stored energy drop to half its initial value?

$$\text{Ans: } 0.69\tau_C, 0.35\tau_C$$

7. Prove that a parallel plate capacitor with square plate of side l making an angle θ with each other produces a capacitance $C = \frac{\epsilon_0 l^2}{d} \left(1 - \frac{\theta}{2d}\right)$ where d is separation of the plate when $\theta = 0$.
8. The parallel plates in a capacitor with a plate area of 8.5 cm^2 and an air filled separation of 3mm are charged by a 6V battery. They are then disconnected from battery and pulled apart to a separation of 8mm. Neglecting fringing, find (a) potential difference between the plates, (b) initial stored energy (c) final stored energy (d) work required to separate the plates. $\text{Ans: (a) } 16v, (b) 45.1pJ (c) 120pJ (d) 75.5 pJ$
9. A 13.5pF parallel plate capacitor has potential difference 12.5V across its plates. The charging battery is now disconnected and a porcelain slab ($k = 6.5$) is slipped between the plates. What is the energy stored before and after the slab is introduced? $\text{Ans: } 1055pJ, 162pJ$
10. What is the capacitance in LC if the maximum charge in capacitor is $1.63 \mu C$ and total energy is $142 \mu J$. $\text{Ans: } 9.35 nC$
11. A wire of resistance 4Ω is stretched to double its original length. What is its resistance now? $\text{Ans: } 16\Omega$
12. A copper wire is stretched to make it 0.1% longer. Find the percentage change in resistance. $\text{Ans: } 0.2\%$
13. The current density across a cylindrical conductor of radius R varies according to the equation $J = J_0 \left(1 - \frac{r}{R}\right)$ where r is the distance from the axis. Thus the current density is maximum J_0 at the axis $r = 0$ and decrease linearly to zero at the surface $r = R$. (a) Calculate the current in terms of J_0 and the conductor's cross sectional area. (b) Suppose that, instead the current density is maximum J_0 at the surface and decrease linearly to zero at the axis, so that $J = \frac{J_0 r}{R}$. Calculate the current.

$$\text{Ans: } \frac{\pi J_0 R^2}{3}, \frac{2\pi J_0 R^2}{3}$$

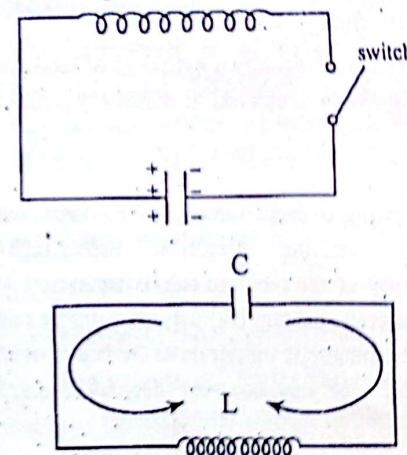
14. Calculate the (a) mean free time and (b) mean free path between collisions for the conduction electron in copper having electron density $8.4 \times 10^{22} \text{ cm}^{-3}$ and resistivity $1.7 \times 10^{-8} \Omega \text{ m}$ if effective speed of electron is $1.6 \times 10^6 \text{ m/s}$.
Ans: $2.5 \times 10^{-14} \text{ s}$, $4.0 \times 10^{-8} \text{ m}$
15. One end of aluminum wire whose diameter is 1.8 mm is welded to one end of copper wire, whose diameter is 1.8 mm. The composite wire carries a steady current of 1.3 A.
 a. What is the current density in each wire?
Ans: 26 A/cm^2 , 51 A/cm^2
 b. Find the drift speed of electron in copper if there is one electron per atom in copper and $M = 63.5$, $N_A = 6.02 \times 10^{23} / \text{mol}$.
Ans: 14 cm/hr
16. The space between two cylinders of radii a and b of length l is filled with a material of resistivity ρ . Show that the radial resistance between the cylinders is given by $R = \frac{\rho}{2\pi l} \ln\left(\frac{b}{a}\right)$
17. According to Bohr model, the hydrogen atom consists of an electron circling a proton at a radius $0.53 \text{ \AA}$. Calculate (i) velocity of the electron (ii) its frequency of circulation (iii) the average current (iv) magnetic dipole moment of the orbit (v) the magnetic induction at the centre of the orbit.
Ans: $2.2 \times 10^6 \text{ m/s}$, $6.6 \times 10^{15} \text{ Hz}$, 1.057 nA , $9.33 \times 10^{-24} \text{ Am}^2$, 12.53 T
18. A parallel plate capacitor with circular plates is charged by the current i . (a) What is the magnitude of $\oint \vec{B} \cdot d\vec{s}$ in terms of μ_0 and i between the plates of $r = \frac{a}{5}$ from centre? (b) What is the induced magnetic field for $r = \frac{a}{5}$ in terms of displacement current?
Ans: $\frac{\mu_0 i}{25}$, $\frac{\mu_0 i d}{10\pi a}$

19. From Kirchhoff's law the current in an RL circuit obeys the differential equation $\frac{dI}{dt} + 5I = 9$ with resistance of 10Ω . Calculate: (a) $I(t)$ (b) inductance (c) time constant (d) maximum current (e) current at time 1s (f) source voltage.
Ans: $\frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right)$, 2 H , 0.2 s , 1.8 A , 1.556 A , 18 V
20. A solenoid has an inductance of 50 H and resistance of 30 ohms . If it is connected to 100 V battery, how long will it take for the current to reach one half of its final equilibrium value?
Ans: 1.2 s
21. The current in an LR circuit builds up to one third of its steady state value in 5 s . find the time constant. *Ans: 12.3 s*
22. A long wire carries a current I of uniform density. Show that the magnetic energy/length within the wire is given by $\frac{\mu_0 I^2}{16\pi}$
23. A long coaxial cable consists of cylinders of radii a and b . The inner cylinder carries a current I , outer cylinder providing the return path (i) Calculate the energy stored in magnetic field for a length of such cable (ii) self inductance of the cable.
Ans: $\frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a}\right)$, $\frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$

- 6.1 Electromagnetic Oscillation
- 6.2 Damped LCR Oscillation
- 6.3 Energy Stored in Electric Field and in Magnetic Field
- 6.4 Damped Electromagnetic Oscillation
- 6.5 Forced Electromagnetic Oscillation and Resonant Frequency
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 - 7.6.1 Energy Density For Electric Field
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- 6.12 Electromagnetic Wave Equation in Non-Conducting Medium (Isotropic Dielectrics)
- 6.13 Intensity of Electromagnetic Wave

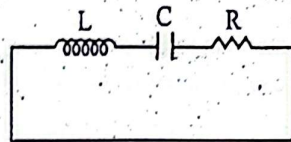
6.1 Electromagnetic Oscillation

The circuit containing both a capacitor C and an inductor L forms an electromagnetic oscillator in which current varies sinusoidally with time resulting an *electromagnetic oscillation*. When a fully charged capacitor is connected with an inductor in series, then the charge of capacitor flows into the inductor, which produces back emf in the inductor. As the capacitor is fully discharged, it again starts to charge by the inductor. During charging, magnetic energy stored in an inductor is transferred to capacitor in the form of electric energy. The process continues resulting an electromagnetic oscillation in the circuit.



6.2 Damped LCR Oscillation

Figure shows LCR circuit. In damped oscillation, amplitude of oscillation decreases exponentially with time due to external resistive force. Here, R stands as resistive factor which decays the current amplitude exponentially with time.



By the principle of conservation of energy, we have

$$U = U_B + U_E$$

$$= \frac{1}{2} Li^2 + \frac{q^2}{2C} \dots\dots\dots(1)$$

Since rate of change of energy is power, we write

$$\text{i.e., } \frac{dU}{dt} = -i^2 R \dots\dots\dots(2)$$

where -ve sign represents that the energy stored, U decreases with time.

From (1),

$$\frac{dU}{dt} = Li \frac{di}{dt} + \frac{q}{C} \frac{dq}{dt} \dots\dots\dots(3)$$

From (2) & (3),

$$Li \frac{di}{dt} + \frac{q}{C} \frac{dq}{dt} = -i^2 R$$

$$\text{or, } Li \frac{d^2 q}{dt^2} + \frac{q}{C} i + i^2 R = 0 \left[\because i = \frac{dq}{dt} \right]$$

$$\text{or, } L \frac{d^2 q}{dt^2} + \frac{q}{C} + iR = 0$$

$$\therefore L \frac{d^2 q}{dt^2} + \frac{dq}{dt} R + \frac{q}{C} = 0$$

This is the required differential equation of damped LCR which can be compared with mechanical damped oscillation.

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

This will give

$$m = L, b = R, K = \frac{1}{C}$$

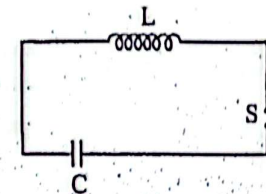
$$\text{Angular frequency } (\omega) = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

$$= \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$\text{Frequency } (f) = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

6.3 Energy Stored in Electric Field and in Magnetic Field

Let Q_0 be the maximum charge stored in capacitor. After the capacitor is fully charged, switch is closed. Then oscillation in circuit can be observed.



The total energy at any instant U in the oscillating circuit is the sum of electric and magnetic energies.

$$\text{i.e., } U = U_{el} + U_{mag}$$

$$= \frac{Q^2}{2C} + \frac{1}{2} Li^2$$

Differentiating with respect to time, we get

$$\frac{dU}{dt} = Li \frac{di}{dt} + \frac{Q}{C} \frac{dQ}{dt}$$

$$i = \frac{dQ}{dt}, \frac{di}{dt} = \frac{d^2 Q}{dt^2}$$

$$\text{or, } \frac{dU}{dt} = L \frac{dQ}{dt} \frac{d^2 Q}{dt^2} + \frac{Q}{C} \frac{dQ}{dt}$$

Conservation of energy implies

$$\frac{dU}{dt} = 0$$

$$\text{or, } L \frac{dQ}{dt} \frac{d^2Q}{dt^2} + \frac{Q}{C} \frac{dQ}{dt} = 0$$

$$\text{or, } L \frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

$$\text{or, } \frac{d^2Q}{dt^2} + \frac{1}{LC} Q = 0 \quad \dots\dots\dots(i)$$

This is the differential equation of LC oscillation:

The differential equation of SHM is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \dots\dots\dots(ii)$$

Expressions (i) and (ii) reveal that LC circuit is an analogy of simple harmonic motion.

Solution of equation (i) is

$$Q = Q_0 \sin(\omega t + \phi).$$

where

$$\omega = \frac{1}{\sqrt{LC}} \text{ is angular frequency.}$$

Energy stored as electric field in capacitor at any time is

$$U_E = \frac{1}{2} \frac{Q^2}{C}$$

$$= \frac{Q_0^2}{2C} \sin^2(\omega t + \phi)$$

The maximum energy stored in electric field is

$$U_{E\max} = \frac{Q_0^2}{2C}$$

Energy stored in inductor as magnetic field is

$$\begin{aligned} U_B &= \frac{1}{2} Li^2 = \frac{1}{2} L \left(\frac{dQ}{dt} \right)^2 = \frac{1}{2} L \omega^2 Q_0^2 \cos^2(\omega t + \phi) \\ &= \frac{1}{2} L \cdot \frac{1}{LC} Q_0^2 \cos^2(\omega t + \phi) \\ &= \frac{Q_0^2}{2C} \cos^2(\omega t + \phi) \end{aligned}$$

The maximum energy stored in magnetic field is

$$U_{B\max} = \frac{Q_0^2}{2C}$$

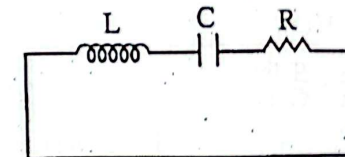
This proves that maximum energy stored in electric field is equal to maximum energy stored in magnetic field.

6.4 Damped Electromagnetic Oscillation

Practically, all oscillations are damping. In case of electromagnetic oscillation, the motion is affected by field produced by oscillator such as electric field and magnetic field, heat produced in resistance, etc. So, these motions are usually damping:

To keep the oscillation continue, we can apply the periodic external force with definite period and frequency. Such an oscillation is called *forced vibration*.

Fig. shows LCR circuit. In damped oscillation, amplitude of oscillation decreases exponentially with time due to external resistive force. Here, R stands as resistive factor which decays the current amplitude exponentially with time.



By the principle of conservation of energy, we have

$$\begin{aligned} U &= U_B + U_E \\ &= \frac{1}{2} Li^2 + \frac{q^2}{2C} \quad \dots\dots\dots(1) \end{aligned}$$

Since rate of change of energy is power, we write

$$\text{i.e., } \frac{dU}{dt} = -i^2 R \quad \dots\dots\dots(2)$$

where -ve sign represents that the energy stored, U decreases with time.

From (1),

$$\frac{dU}{dt} = Li \frac{di}{dt} + \frac{q}{C} \frac{dq}{dt} \dots\dots(3)$$

From equations (2) & (3),

$$Li \frac{di}{dt} + \frac{q}{C} \frac{dq}{dt} = -i^2 R$$

$$\text{or, } Li \frac{d^2 q}{dt^2} + \frac{q}{C} i + i^2 R = 0 \left[\because i = \frac{dq}{dt} \right]$$

$$\text{or, } L \frac{d^2 q}{dt^2} + \frac{q}{C} + iR = 0$$

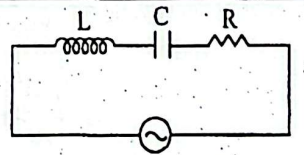
$$\therefore L \frac{d^2 q}{dt^2} + \frac{dq}{dt} R + \frac{q}{C} = 0$$

This is the required differential equation of damped electromagnetic oscillation.

The solution of above differential equation is

$$q = q_0 \sin(\omega t - \phi)$$

6.5 Forced Electromagnetic Oscillation and Resonant Frequency



A circuit containing resistance (R), inductance (L), and capacitance (C) driven by external source is called *forced electromagnetic oscillation*.

If $E = E_0 \sin \omega t$ be the external emf applied to the circuit, then, from Kirchoff's law,

$$L \frac{di}{dt} + \frac{q}{C} + iR = E_0 \sin \omega t \dots\dots\dots(1)$$

$$\text{or, } \frac{d}{dt} \left(\frac{dq}{dt} \right) + \frac{dq}{dt} \frac{R}{L} + \frac{q}{LC} = \left(\frac{E_0}{L} \right) \sin \omega t$$

$$\text{or, } \frac{d^2 q}{dt^2} + \left(\frac{R}{L} \right) \frac{dq}{dt} + \frac{q}{LC} = \frac{E_0}{L} \sin \omega t \dots\dots\dots(2)$$

This is the differential equation of forced electromagnetic oscillation. Solution of this equation is

$$q = \frac{\frac{E_0}{L}}{\sqrt{\left(\frac{1}{LC} - \omega^2 \right)^2 + \left(\frac{\omega R}{L} \right)^2}} \sin(\omega t - \phi)$$

From equation (2), angular frequency due to force is

$$\omega^2 = \omega_0^2 - 2K^2$$

$$\therefore \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

If the observed frequency of oscillation is equal to natural frequency, then resonance will occur.

$$\text{Using, } q = \frac{\frac{E_0}{L}}{\sqrt{\left(\frac{1}{LC} - \omega^2 \right)^2 + \left(\frac{\omega R}{L} \right)^2}} \sin(\omega t - \phi)$$

$$\text{or, } i = \frac{dq}{dt}$$

$$= \frac{\frac{E_0 \omega}{L}}{\sqrt{\left(\frac{1}{LC} - \omega^2 \right)^2 + \left(\frac{\omega R}{L} \right)^2}} \cos(\omega t - \phi)$$

For maximum amplitude of current,

$$\left(\frac{1}{LC} - \omega^2 \right)^2 = 0$$

$$\text{or, } \omega^2 = \frac{1}{LC}$$

$$\text{or, } \omega = \frac{1}{\sqrt{LC}}$$

$$\therefore \text{Resonant frequency } (f) = \frac{\omega}{2\pi}$$

$$= \frac{1}{2\pi\sqrt{LC}}$$

6.6 Sharpness of Resonance

Sharpness of resonance is defined as the ratio of the bandwidth of the circuit to the resonant frequency.

$$\text{Sharpness of resonance} = \frac{f_2 - f_1}{f_0}$$

$$= \frac{\text{Bandwidth}}{f_0} = \frac{1}{Q_0}$$

Sharpness of resonance gives the selectivity of the circuit. As the bandwidth decreases, selectivity of the circuit increases.

6.7 Differences Between Mechanical Oscillation and Electromagnetic Oscillation

Mechanical oscillation and electromagnetic oscillation both are simple harmonic motion.

In mechanical oscillation, mass of an object oscillates, whereas in e.m. oscillation, negative or positive charge oscillates about mean position. Mechanical oscillation can be represented by $\frac{d^2y}{dt^2} + \omega^2y = 0$, whereas e.m. oscillation can be represented by $\frac{d^2q}{dt^2} + \omega^2q = 0$. Displacement in mechanical oscillation is analogous to charge in e.m. oscillation.

7.6.1 Energy Density For Electric Field

Total energy stored in capacitor is given by

$$E = \frac{1}{2} CV^2$$

For parallel plate capacitor, $C = \frac{\epsilon_0 A}{d}$

$$\text{So, } E = \frac{1}{2} \frac{\epsilon_0 A}{d} V^2$$

Now, energy density, $u_E = \frac{\text{Total energy}}{\text{Volume}}$

$$= \frac{\frac{1}{2} \frac{\epsilon_0 A}{d} V^2}{\text{Volume}}$$

$$= \frac{\frac{1}{2} \frac{\epsilon_0 A}{d} V^2}{Ad}$$

$$= \frac{1}{2} \epsilon_0 \left(\frac{V}{d}\right)^2$$

$$= \frac{1}{2} \epsilon_0 E^2$$

$$\therefore u_E \propto E^2$$

7.6.2 Energy Density For Magnetic Field

Total energy stored in inductor is

$$E = \frac{1}{2} LI^2$$

Now, energy density, $u_B = \frac{\text{Total energy}}{\text{Volume}}$

$$= \frac{\frac{1}{2} LI^2}{Al}$$

$$= \frac{\frac{1}{2} I^2}{A} \frac{L}{l}$$

$$= \frac{\frac{1}{2} I^2}{A} (\mu_0 n^2 A)$$

$$= \frac{1}{2\mu_0} (\mu_0 n I)^2$$

$$= \frac{1}{2\mu_0} B^2$$

$$\therefore u_B \propto B^2$$

Hence, energy per unit volume in electric field and magnetic field are proportional to the square of their fields.

6.8 Maxwell Equations

Maxwell's equations are the unification of electric and magnetic field in nature. Various properties of electromagnetic waves can be understood from Maxwell's equations. There are four relations in electromagnetic field which are known as Maxwell's equations.

i. Gauss' Law for Electrostatics

$$\oint \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0} \dots\dots\dots(1)$$

According to first equation, total electric flux passing through closed surface is equal to $\frac{1}{\epsilon_0}$ times the total charge enclosed.

$$\text{or, } \oint \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} \int \rho \, dv \quad [\because q = \int \rho \, dv]$$

Using divergence theorem,

$$\oint \vec{E} \cdot d\vec{s} = \int \nabla \cdot \vec{E} \, dv$$

$$\text{or, } \int \nabla \cdot \vec{E} \, dv = \frac{1}{\epsilon_0} \int \rho \, dv$$

$$\text{or, } \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \text{ which is differential form of Maxwell's equation.}$$

ii. Gauss' Law for Magnetism

$$\oint \vec{B} \cdot d\vec{s} = 0$$

Since magnetic lines of force are either closed or go off to infinity, the number of magnetic lines of force entering any

arbitrary closed surface is equal to lines of force leaving from it. It means the flux of magnetic induction $\vec{B}$ across any closed surface is zero.

$$\text{or, } \int \nabla \cdot \vec{B} \, dv = 0$$

As the surface bounding the volume is arbitrary, therefore this equation holds only if the integral vanish

$$\text{i.e., } \nabla \cdot \vec{B} = 0$$

which is differential form of Maxwell's second equation.

iii. Faraday's Law of Electromagnetic Induction

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t}$$

When magnetic field varies with time, an electric field will be produced. Here, electromotive force around a closed path is equal to the time derivative of the magnetic flux through any closed surface.

Using Stoke's theorem to above equation,

$$\oint (\nabla \times \vec{E}) \cdot d\vec{s} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{s}$$

$$\text{i.e., } \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

which is differential form of Maxwell's 3rd equation.

iv. Ampere's Law as Extended by Maxwell

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(i + \epsilon_0 \frac{\partial \phi_E}{\partial t} \right)$$

As soon as electric field is applied between plates of capacitor, electrons of dielectric undergo a displacement within atom. This electronic movement constitutes displacement current. Hence, the total current through the circuit will be sum of displacement current and conduction current.

$$\text{or, } (\nabla \times \vec{B}) \cdot d\vec{s} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \cdot d\vec{s}$$

$$\therefore \nabla \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right]$$

This is Maxwell's fourth equation in differential form.

6.8.1 Maxwell's Equations in Free Space and Dielectric Medium

⇒ Maxwell's equations are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \dots\dots\dots (i)$$

$$\nabla \cdot \vec{B} = 0 \dots\dots\dots (ii)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \dots\dots\dots (iii)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \dots\dots\dots (iv)$$

In free space $\rho = 0$, $\vec{J} = 0$.

i. Maxwell's Equation in Free Space

$$\nabla \cdot \vec{E} = 0 \dots\dots\dots (i)$$

$$\nabla \cdot \vec{B} = 0 \dots\dots\dots (ii)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \dots\dots\dots (iii)$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \dots\dots\dots (iv)$$

ii. Maxwell's Equation in Dielectric Medium

$$\nabla \cdot \vec{E} = 0 \dots\dots\dots (i)$$

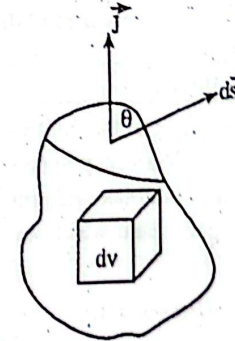
$$\nabla \cdot \vec{B} = 0 \dots\dots\dots (ii)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \dots\dots\dots (iii)$$

$$\nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} \dots\dots\dots (iv)$$

6.9 Charge Conservation Theorem

According to the principle of conservation of charge, net amount of charge in an isolated system must remain constant. If the charge varies with time, the net charge leaving the volume crossing the boundary surface equals the net rate of flow of charge entering the volume.



Consider a surface element $d\vec{s}$ of surface s enclosing a volume v with $\vec{J}$ as the current density directed as shown.

$\vec{J} \cdot d\vec{s}$ represents charge per unit time leaving the volume v across ds . Therefore, $\int \vec{J} \cdot d\vec{s}$ represents charge per unit time leaving the volume v crossing the entire surface $\vec{s}$. Let q be instantaneous charge within volume.

Then, $-\frac{dq}{dt}$ measures the rate of decrease of charge within the volume.

Principle of conservation of charge gives

$$\int \vec{J} \cdot d\vec{s} = \frac{-dq}{dt} = -\frac{d}{dt} \int \rho dv$$

Gauss' divergence theorem is $\int \vec{A} \cdot d\vec{s} = \int_{vol} \nabla \cdot \vec{A} dv$

$$\text{or, } \int (\nabla \cdot \vec{J}) dv = -\int \frac{\partial \rho}{\partial t} dv$$

$$\text{or, } \int \left((\nabla \cdot \vec{J}) + \frac{\partial \rho}{\partial t} \right) dv = 0$$

Since the volume is arbitrary, integrand must vanish.

$$\text{Therefore, } \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

which is the principal of conservation of charge or continuity equation.

6.10 Electromagnetic Wave Equation in Free Space

Maxwell's equations are given by

$$\left. \begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \dots\dots\dots (i) \\ \nabla \cdot \vec{B} &= 0 \dots\dots\dots (ii) \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \dots\dots\dots (iii) \\ \nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \dots\dots\dots (iv) \end{aligned} \right\} \dots\dots (1)$$

In free $\rho = 0$, $\vec{J} = 0$. Therefore, Maxwell's equations reduce to

$$\left. \begin{aligned} \nabla \cdot \vec{E} &= 0 \dots\dots\dots (i) \\ \nabla \cdot \vec{B} &= 0 \dots\dots\dots (ii) \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \dots\dots\dots (iii) \\ \nabla \times \vec{B} &= \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \dots\dots\dots (iv) \end{aligned} \right\} \dots\dots (2)$$

Using an identity,

$$\nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - (\nabla \cdot \nabla) \vec{E}$$

$$\text{or, } \nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E}$$

Using equation 2 (iii),

$$\nabla \times -\frac{\partial \vec{B}}{\partial t} = -\nabla^2 \vec{E}$$

$$\text{or, } \frac{\partial (\nabla \times \vec{B})}{\partial t} = \nabla^2 \vec{E}$$

From equation 2 (iv),

$$\frac{\partial \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)}{\partial t} = \nabla^2 \vec{E}$$

$$\text{or, } \nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\text{or, } \nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\text{or, } \nabla^2 \vec{E} = \frac{1}{c^2} \frac{d^2 \vec{E}}{dt^2} \dots\dots\dots (3)$$

Again using an identity,

$$\nabla \times (\nabla \times \vec{B}) = \nabla (\nabla \cdot \vec{B}) - (\nabla \cdot \nabla) \vec{B}$$

$$\text{or, } \nabla \times (\nabla \times \vec{B}) = -\nabla^2 \vec{B}$$

From equation 2 (iv),

$$\nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = -\nabla^2 \vec{B}$$

$$\text{or, } \mu_0 \epsilon_0 \frac{\partial (\nabla \times \vec{E})}{\partial t} = -\nabla^2 \vec{B}$$

From equation 2 (iii),

$$-\mu_0 \epsilon_0 \frac{\partial \left(\frac{\partial \vec{B}}{\partial t} \right)}{\partial t} = -\nabla^2 \vec{B}$$

$$\text{or, } \nabla^2 \vec{B} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

$$\text{or, } \nabla^2 \vec{B} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

$$\text{or, } \nabla^2 \vec{B} = \frac{1}{c^2} \frac{d^2 \vec{B}}{dt^2} \dots\dots\dots (4)$$

Equations (3) and (4) represent the standard equation of wave travelling with velocity

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{8.85 \times 10^{-12} \times 4\pi \times 10^{-7}}} \approx 3 \times 10^8 \text{ m/s}$$

Hence the electromagnetic wave travels with the velocity of light.

The solutions of equation (3) and (4) are given by

$$E = E_0 e^{-i(\omega t - \vec{k} \cdot \vec{r})}$$

$$B = B_0 e^{-i(\omega t - \vec{k} \cdot \vec{r})}$$

6.11 Maxwell's Electromagnetic Wave Equation in Conducting Medium

Maxwell's equations are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \dots\dots\dots (i)$$

$$\nabla \cdot \vec{B} = 0 \dots\dots\dots (ii)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \dots\dots\dots (iii)$$

$$\nabla \times \vec{B} = \mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \dots\dots\dots (iv)$$

} (1)

Let the medium be linear and isotropic characterized by permittivity ϵ , permeability μ , and conductivity σ , but not any charge or any current other than that determined by Ohm's law.

$$\text{i.e., } \rho = 0, \vec{J} = \sigma \vec{E}$$

Hence the Maxwell's equations become

$$\nabla \cdot \vec{E} = 0 \dots\dots\dots (i)$$

$$\nabla \cdot \vec{B} = 0 \dots\dots\dots (ii)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \dots\dots\dots (iii)$$

$$\nabla \times \vec{B} = \mu \sigma \vec{E} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \dots\dots\dots (iv)$$

} (2)

Using an identity,

$$\nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - (\nabla \cdot \nabla) \vec{E}$$

$$\text{or, } \nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E}$$

Using equation 2 (iii),

$$\nabla \times -\frac{\partial \vec{B}}{\partial t} = -\nabla^2 \vec{E}$$

$$\text{or, } \frac{\partial (\nabla \times \vec{B})}{\partial t} = \nabla^2 \vec{E}$$

From equation 2 (iv),

$$\frac{\partial \left(\mu \sigma \vec{E} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \right)}{\partial t} = \nabla^2 \vec{E}$$

$$\text{or, } \nabla^2 \vec{E} - \sigma \mu \frac{\partial \vec{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\text{or, } \nabla^2 \vec{E} - \sigma \mu \frac{\partial \vec{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \dots\dots\dots (3)$$

Again using an identity,

$$\nabla \times (\nabla \times \vec{B}) = \nabla (\nabla \cdot \vec{B}) - (\nabla \cdot \nabla) \vec{B}$$

$$\text{or, } \nabla \times (\nabla \times \vec{B}) = -\nabla^2 \vec{B}$$

From equation 2 (iv),

$$\nabla \times \left(\mu \sigma \vec{E} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \right) = -\nabla^2 \vec{B}$$

$$\text{or, } \mu \sigma (\nabla \times \vec{E}) + \mu \epsilon \frac{\partial (\nabla \times \vec{E})}{\partial t} = -\nabla^2 \vec{B}$$

From equation 2 (iii),

$$-\mu \sigma \frac{\partial \vec{B}}{\partial t} - \mu \epsilon \frac{\partial \left(\frac{\partial \vec{B}}{\partial t} \right)}{\partial t} = -\nabla^2 \vec{B}$$

$$\text{or, } \nabla^2 \vec{B} - \mu \sigma \frac{\partial \vec{B}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

$$\text{or, } \nabla^2 \vec{B} - \mu \sigma \frac{\partial \vec{B}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \quad \dots\dots(4)$$

The equations (3) and (4) represent wave equations for electromagnetic field in a homogenous isotropic conducting medium, satisfying wave equation of the form

$$\nabla^2 \psi - \mu \sigma \frac{\partial \psi}{\partial t} - \mu \epsilon \frac{\partial^2 \psi}{\partial t^2} = 0 \quad \dots\dots(5)$$

The solutions of equation (3), (4) and (5) are given by

$$E = E_0 e^{-i(\omega t - \vec{k} \cdot \vec{r})}$$

$$B = B_0 e^{-i(\omega t - \vec{k} \cdot \vec{r})}$$

$$\psi = \psi_0 e^{-i(\omega t - \vec{k} \cdot \vec{r})}$$

6.12 Electromagnetic Wave Equation in Non-Conducting Medium (Isotropic Dielectrics)

A non-conducting medium which has same properties in all directions is called an isotropic dielectric.

Maxwell's equations are:

$$\left. \begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon} \quad \dots\dots\dots (i) \\ \nabla \cdot \vec{B} &= 0 \quad \dots\dots\dots (ii) \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \quad \dots\dots\dots (iii) \\ \nabla \times \vec{B} &= \mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \quad \dots\dots\dots (iv) \end{aligned} \right\} \quad \dots\dots\dots (1)$$

In an isotropic dielectric, $\rho = 0$; $\vec{J} = 0$.

Therefore Maxwell's equations become

$$\left. \begin{aligned} \nabla \cdot \vec{E} &= 0 \quad \dots\dots\dots (i) \\ \nabla \cdot \vec{B} &= 0 \quad \dots\dots\dots (ii) \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \quad \dots\dots\dots (iii) \\ \nabla \times \vec{B} &= \mu \epsilon \frac{\partial \vec{E}}{\partial t} \quad \dots\dots\dots (iv) \end{aligned} \right\} \quad \dots\dots\dots (2)$$

Using an identity,

$$\nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - (\nabla \cdot \nabla) \vec{E}$$

$$\text{or, } \nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E}$$

From equation 2 (iii),

$$\nabla \times -\frac{\partial \vec{B}}{\partial t} = -\nabla^2 \vec{E}$$

$$\text{or, } \frac{\partial (\nabla \times \vec{B})}{\partial t} = \nabla^2 \vec{E}$$

From equation 2 (iv),

$$\frac{\partial (\mu \epsilon \frac{\partial \vec{E}}{\partial t})}{\partial t} = \nabla^2 \vec{E}$$

$$\text{or, } \nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \dots\dots\dots (3)$$

Again using an identity,

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - (\nabla \cdot \nabla) \vec{B}$$

$$\text{or, } \nabla \times (\nabla \times \vec{B}) = -\nabla^2 \vec{B}$$

Using equation 2 (iv),

$$\text{or, } \nabla \times \mu\epsilon \frac{\partial \vec{E}}{\partial t} = -\nabla^2 \vec{B}$$

$$\text{or, } \mu\epsilon \frac{\partial(\nabla \times \vec{E})}{\partial t} = -\nabla^2 \vec{B}$$

Using equation 2 (iii),

$$\mu\epsilon \frac{\partial}{\partial t} \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\nabla^2 \vec{B}$$

$$\text{or, } \nabla^2 \vec{B} = \mu\epsilon \left(\frac{\partial^2 \vec{B}}{\partial t^2} \right)$$

$$\text{or, } \nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \dots\dots\dots (4)$$

The scalar wave equations (3) and (4) are of the form

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

$$\text{with velocity } v = \frac{1}{\sqrt{\mu\epsilon}}$$

Also we have $\mu = \mu_r \mu_0$ and $\epsilon = \epsilon_r \epsilon_0$

where μ_r and ϵ_r are relative permeability and relative permittivity of medium respectively.

$$\begin{aligned} \text{Now, } v &= \frac{1}{\sqrt{\mu_r \epsilon_r \mu_0 \epsilon_0}} \\ &= \frac{c}{\sqrt{\mu_r \epsilon_r}} \end{aligned}$$

Since $\mu_r > 1$ and $\epsilon_r > 1$, the speed of electromagnetic wave in isotropic dielectric medium is less than the speed of electromagnetic wave in free space.

6.13 Intensity of Electromagnetic Wave

The average value of $\vec{S}$ gives intensity of electromagnetic wave.

$$I = S_{\text{avg}} = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2\mu_0 c}$$

$$\text{or, } I = \frac{E_{\text{rms}}^2}{\mu_0 c} \quad \left[\because E_{\text{rms}}^2 = \frac{E_0^2}{2} \right]$$

$$\text{Hence, } I \propto E_{\text{rms}}^2$$

SOLUTION TO EXAM PROBLEMS AND IMPORTANT QUESTIONS

1. A 1.3mH inductor in LC circuit stores a maximum energy of 17μJ. what is the value of peak current?

$$\Rightarrow \text{Inductance}(L) = 1.3\text{mH} = 1.3 \times 10^{-3} \text{ H}$$

$$\text{Energy } (E_{\max}) = 17\mu\text{J} = 17 \times 10^{-6} \text{ J}$$

$$\text{We know, } E_{\max} = \frac{1}{2} L I_{\max}^2$$

$$\text{or, } I_{\max} = \sqrt{\frac{2E_{\max}}{L}}$$

$$= \sqrt{\frac{2 \times 17 \times 10^{-6}}{1.3 \times 10^{-3}}} = 0.15 \text{ A}$$

2. Deduce the frequency and quality factor for a circuit with $L=2 \text{ mH}$, $C = 5 \mu\text{F}$, and $R = 0.2 \Omega$

$\Rightarrow$ We know,

$$\text{Frequency } (f) = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$\text{Or, } f = \frac{1}{2\pi} \sqrt{\frac{1}{2 \times 10^{-3} \times 5 \times 10^{-6}} - \left(\frac{0.2}{2 \times 2 \times 10^{-3}}\right)^2}$$

$$= 1591.53 \text{ Hz}$$

Now,

$$\text{Quality factor } (Q) = \frac{L\omega_0}{R}$$

$$= \frac{L(2\pi f)}{R}$$

$$= \frac{2 \times 10^{-3} \times 2\pi \times 1591.53}{0.2}$$

$$\therefore Q = 100.$$

3. An LC circuit is converted in to LCR circuit inserting a resistance of 10Ω . Calculate the percentage change in frequency in this conversion. Given: inductance = 10 mH and capacitance = $10 \mu\text{F}$. [2014 Fall]

$$\Rightarrow \text{Resistance } (R) = 10 \Omega$$

$$\text{Inductance } (L) = 10 \text{ mH}$$

$$\text{Capacitance } (C) = 10 \mu\text{F}$$

$$\text{Frequency of LC oscillation } (f) = \frac{1}{2\pi\sqrt{LC}}$$

$$= 503.3 \text{ Hz}$$

$$\text{Frequency of LCR oscillation } (f') = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$\text{or, } f' = \frac{1}{2\pi} \sqrt{\frac{1}{10 \times 10^{-3} \times 10 \times 10^{-6}} - \left(\frac{10}{2 \times 10 \times 10^{-3}}\right)^2}$$

$$= 496.96 \text{ Hz}$$

Hence the percentage change in frequency is

$$\frac{f - f'}{f} \times 100 = 1.26\%$$

4. A 40mH inductor and 1000μF capacitor form oscillating circuit. What is the peak value of current if the initial charge is 40μC? [2016 Spring]

$\Rightarrow$ We know the maximum energy stored by capacitor and

$$\text{inductor is } \frac{q^2}{2C} = \frac{1}{2} L I_{\max}^2$$

$$\text{or, } \frac{(40 \times 10^{-6})^2}{2 \times 1000 \times 10^{-6}} = \frac{1}{2} \times 40 \times 10^{-3} \times I_{\max}^2$$

$$\therefore I_{\max} = 6.32 \text{ A}$$

5. If 10mH inductor and two capacitors of 5μF and 2μF are given, find the two resonant frequencies that can be obtained by connecting these elements in different ways. [2018 Spring]

⇒ Inductance (L) = 10mH

Capacitance (C₁) = 5μF, C₂ = 2μF

We know,

Equivalent capacitance in series parallel combination of capacitor (C) = C₁ + C₂ = 7μF

Resonance frequency in this case is

$$(f) = \frac{1}{2\pi\sqrt{LC}} = 601.55\text{Hz}$$

$$\text{Again, equivalent capacitance in series } C = \frac{C_1 \times C_2}{C_1 + C_2} = 1.43\mu\text{F}$$

Again, Resonance frequency in this case is

$$(f) = \frac{1}{2\pi\sqrt{LC}} = 1330.92\text{ Hz}$$

6. A circuit has L = 10 mH and C = 10 μF. How much resistance should be added to circuit so that the frequency of oscillation will be 1% less than that of free LC oscillation? [2018 Fall]

⇒ Inductance (L) = 10 mH

Capacitance (C) = 10 μF

Damped frequency = 0.99 Undamped frequency

$$\text{or, } \frac{1}{2\pi\sqrt{LC}} - \left(\frac{R}{2L}\right)^2 = \frac{0.99}{2\pi\sqrt{LC}}$$

$$\text{or, } \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \frac{0.99}{\sqrt{LC}}$$

$$\text{or, } \frac{R^2}{4L^2} = \frac{1}{LC} - \frac{0.99^2}{LC}$$

$$\therefore R = 8.9 \Omega$$

7. The maximum electric field 10 m from an isotropic point source of light is 2 V/m. What are (a) the maximum value of the magnetic field and (b) the average intensity of the light there? (c) what is the power of the source?

⇒ Electric field (E₀) = 2 V/m

Distance from source (r) = 10 m

Maximum electric field (B₀) = ?

Average intensity (I) = ?

Power of the source (P) = ?

We have,

$$E_0 = B_0 c \Rightarrow B_0 = \frac{E_0}{c} = \frac{2}{3 \times 10^8} = 0.66 \times 10^{-8} \text{ T}$$

Average intensity is

$$I = \frac{1}{2} c \epsilon_0 E_0^2 = \frac{1}{2} \times 3 \times 10^8 \times 8.85 \times 10^{-12} \times (2)^2 = 5.31 \times 10^{-3} \text{ watt/m}^2$$

Power of the source (P) = I × A

$$= I \times 4\pi r^2$$

$$= 5.31 \times 10^{-3} \times 4 \times 3.14 \times (10)^2$$

$$= 6.66 \text{ watt}$$

8. A radio wave transmits 25 W/m² of power per unit area. A flat surface area is perpendicular to the direction of propagation of the wave. Calculate the radiation pressure on it and maximum electric and magnetic field associated with the wave.

⇒ The radiation pressure for total absorption of radiation is

$$P = \frac{I}{c}$$

$$= \frac{25}{3 \times 10^8}$$

$$= 83.4 \times 10^{-19} \text{ Pa}$$

Using,

$$I = \frac{E_0^2}{2\mu_0 c} = \frac{c B_0^2}{2\mu_0}$$

Maximum electric field is

$$\begin{aligned} E_0 &= \sqrt{2\mu_0 c I} \\ &= \sqrt{2 \times 4\pi \times 10^{-7} \times 3 \times 10^8 \times 25} \\ &= 137.3 \text{ V/m} \end{aligned}$$

Maximum value of magnetic field is

$$\begin{aligned} B_0 &= \sqrt{\frac{2\mu_0 I}{c}} \\ &= \sqrt{\frac{2 \times 4\pi \times 10^{-7} \times 25}{3 \times 10^8}} \\ &= 4.57 \times 10^{-7} \text{ Tesla} \end{aligned}$$

9. At some distance from transmitter of radio station, the magnetic field of electromagnetic wave emitted by radio station is found to be $1.6 \times 10^{-4} \text{ T}$. If frequency of broadcast is 1020 KHz then find speed, wavelength and maximum electric field of electromagnetic wave.

[2017 Spring, 2014 Spring]

⇒ The speed of electromagnetic wave is equal to the speed of light $c = 3 \times 10^8 \text{ m/s}$.

$$\text{Wavelength } (\lambda) = \frac{c}{f}$$

$$\begin{aligned} &= \frac{3 \times 10^8}{1020 \times 10^3} \\ &= 294.12 \text{ m} \end{aligned}$$

$$\text{Maximum electric field is } (E_0) = c B_0 = 4.8 \times 10^4 \text{ V/m}.$$

10. Show that ratio of electric field strength and magnetic field strength is equal to speed of light hence calculate maximum value of magnetic field and average intensity of light, If the maximum electric field at a distance of 10cm from an isotropic light source is 2V/m. [2017 Fall]

$$\Rightarrow \text{Maxwell's third equation is } \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{or, } \frac{dE_y}{dx} = -\frac{dB_z}{dt} \dots\dots\dots(i)$$

Solution of electromagnetic wave, is

$$E = E_0 \sin(\omega t - kx) \dots\dots\dots(ii) \text{ and}$$

$$B = B_0 \sin(\omega t - kx) \dots\dots\dots(iii)$$

Differentiating eq (ii) with respect to x and eq (iii) with respect to t

$$\frac{dE_y}{dx} = -k E_0 \sin(\omega t - kx) \dots\dots\dots(iv)$$

$$\frac{dB_z}{dt} = \omega B_0 \sin(\omega t - kx) \dots\dots\dots(v)$$

Now, equation (i) becomes,

$$-k E_0 \sin(\omega t - kx) = -\omega B_0 \sin(\omega t - kx)$$

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c \text{ proved.}$$

Second part

$$\text{Maximum magnetic field } (B_0) = \frac{E_0}{c}$$

$$= \frac{2}{3 \times 10^8} = 6.6 \text{ nT}$$

$$\text{Average intensity of light is } I = \frac{E_0^2}{2\mu_0 c}$$

$$\begin{aligned} &= \frac{2^2}{2 \times 4\pi \times 10^{-7} \times 3 \times 10^8} \\ &= 1.32 \mu \text{ Watt/m}^2 \end{aligned}$$

11. The sun delivers about 10^3 W/m^2 of energy to surface through EM radiation.

Calculate:

- the total power incident on a roof of dimensions $8\text{m} \times 20\text{m}$.
- radiation pressure and force exerted on the roof, assuming roof is perfect absorber.

$$\Rightarrow \text{Intensity of radiation (S)} = 10^3 \text{ W/m}^2$$

$$\text{We know the relation, Intensity (S)} = \frac{\text{Power (P)}}{\text{Area (A)}}$$

$$\text{Power (P)} = S \times A$$

$$= 10^3 \times 160 = 1.6 \times 10^5 \text{ Watt.}$$

$$\text{Radiation pressure (P}_r\text{)} = \frac{\text{Intensity (S)}}{c}$$

$$= \frac{10^3}{3 \times 10^8}$$

$$= 3.33 \times 10^{-6} \text{ N/m}^2$$

$$\text{Force exerted (F)} = \text{Pressure} \times \text{Area}$$

$$= 3.33 \times 10^{-6} \times 160$$

$$= 5.33 \times 10^{-4} \text{ N}$$

12. Sunlight strikes the earth outside its atmosphere with an intensity of $2 \text{ Cal/cm}^2\text{-min}$. Calculate the magnitude of electric and magnetic fields.

$$\Rightarrow \text{Intensity of solar radiation (S)} = 2 \text{ Cal/cm}^2\text{-min}$$

$$= \frac{2 \times 4.2}{(10^{-2})^2 \times 60}$$

$$= 1400 \text{ J/m}^2\text{-sec}$$

$$\text{We know, } S = \frac{E_0^2}{2\mu_0 c}$$

$$\begin{aligned} \therefore E_0 &= \sqrt{2S\mu_0 c} \\ &= \sqrt{2 \times 1400 \times 4\pi \times 10^{-7} \times 3 \times 10^8} \\ &= 1027.4 \text{ V/m} \end{aligned}$$

Again for magnetic field,

$$S = \frac{B_0^2 c}{2\mu_0}$$

$$\therefore B_0 = \sqrt{\frac{2S\mu_0}{c}}$$

$$\begin{aligned} &= \sqrt{\frac{2 \times 1400 \times 4\pi \times 10^{-7}}{3 \times 10^8}} \\ &= 3.42 \times 10^{-6} \text{ Tesla} \end{aligned}$$

13. Using Maxwell's equation, prove that $c = \frac{E_0}{B_0}$ Where the symbols carry their usual meaning. [2021 Fall]

$$\Rightarrow \text{Maxwell's third equation is } \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{or, } \frac{dE_y}{dx} = -\frac{dB_z}{dt} \dots\dots\dots(i)$$

Solution of electromagnetic wave, is

$$E = E_0 \sin(\omega t - kx) \dots\dots\dots(ii) \text{ and}$$

$$B = B_0 \sin(\omega t - kx) \dots\dots\dots(iii)$$

Differentiating equation (ii) with respect to x and equation (iii) with respect to t,

$$\frac{dE_y}{dx} = -kE_0 \sin(\omega t - kx) \dots\dots\dots(iv)$$

$$\frac{dB_z}{dt} = \omega B_0 \sin(\omega t - kx) \dots\dots\dots(v)$$

Now, equation (i) becomes,

$$-kE_0 \sin(\omega t - kx) = -\omega B_0 \sin(\omega t - kx)$$

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c \quad \text{proved.}$$

14. If a test charge revolves round a circular path of radius 8.5 cm where the magnetic force increases at a steady rate 0.13 T/s, calculate the magnitude of induced electric field at a point 12.5 cm.

⇒ Here, radius of circular path (R) = 8.5 cm
 $= 0.085 \text{ m}$
 Rate of change of magnetic field, $\frac{dB}{dt} = 0.13 \text{ T/s}$
 Distance of point from centre (r) = 12.5 cm
 $= 0.125 \text{ m}$

We know from Maxwell's equation,

$$\oint \vec{E} \cdot d\vec{l} = \frac{d\phi_B}{dt} \quad (\text{ignoring negative sign})$$

$$E 2\pi r = \frac{d(BA)}{dt} = \pi R^2 \frac{dB}{dt}$$

$$\text{or, } E = \frac{R^2}{2r} \frac{dB}{dt} = \frac{0.085^2 \times 0.13}{2 \times 0.125}$$

$$\therefore E = 3.757 \times 10^{-3} \text{ Vm}^{-1}$$

15. A wire of length L carries a current I. If the wire is formed in to a circular coil, then the maximum torque in a given magnetic field B developed for a single turn is $\tau = \left(\frac{1}{4\pi}\right) L^2 IB$.

⇒ Let r be the radius of circular coil. So,

$$2\pi r = L$$

$$\text{or, } r = \frac{L}{2\pi}$$

Torque produced by magnetic field on current carrying coil is

$$\tau = BINA = BI(\pi r^2) \quad ; \quad \text{for single turn, } N=1$$

$$\text{or, } \tau = \pi BI \frac{L^2}{4\pi}$$

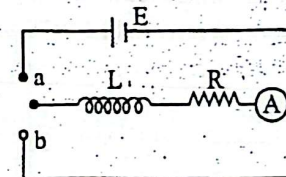
$$\therefore \tau = \left(\frac{1}{4\pi}\right) L^2 IB \quad \text{proved.}$$

16. An inductance L is connected to a battery of emf E through a resistance. Show that the potential difference across the inductance after time t is $V_L = \epsilon e^{\left(\frac{-R}{L}\right)t}$. At what time is the potential difference across the inductance equal to that across the resistance such that $i = \frac{i_0}{2}$.

⇒ The growth of the current through RL circuit is

$$i = i_0 (1 - e^{-t/\tau})$$

$$= i_0 \left(1 - e^{-\frac{Rt}{L}}\right)$$



Potential difference across the inductor is

$$V_L = L \frac{di}{dt}$$

$$= -L i_0 \left(\frac{-R}{L}\right) e^{-\frac{Rt}{L}}$$

$$= i_0 R e^{-\frac{Rt}{L}}$$

$$\therefore V_L = \epsilon e^{-\frac{Rt}{L}} \quad \text{proved.}$$

When the potential difference across the inductor is equal to that of resistance, we write

$$V_L = V_R$$

$$\text{or, } \epsilon e^{\frac{-Rt}{L}} = iR$$

$$\therefore e^{\frac{Rt}{L}} = \frac{\epsilon}{iR}$$

According to question, $i = \frac{i_0}{2}$

$$\text{or, } \frac{i_0}{i} = 2$$

$$\text{or, } e^{\frac{Rt}{L}} = 2$$

$$\text{or, } \frac{Rt}{L} = \ln 2$$

$$\text{or, } t = \frac{L}{R} \ln 2$$

$$\therefore t = 0.693 \frac{L}{R} \text{ sec}$$

17. What is the initial rate of increase of current and final saturation current with $L = 15 \text{ mH}$, $R = 24 \Omega$ and emf = 10 V [2015 Fall]

$$\Rightarrow \text{Final saturation current } (I_0) = \frac{\epsilon}{R}$$

$$= \frac{10}{24}$$

$$= 0.416 \text{ A}$$

$$\text{Inductive time constant } (\tau) = \frac{L}{R}$$

$$= \frac{15}{24} = 0.625 \text{ sec.}$$

$$\text{Current in the circuit is } I = I_0 (1 - e^{-t/\tau})$$

$$\text{Rate of increase of current is } \frac{dI}{dt} = \frac{I_0}{\tau} e^{-t/\tau}$$

Therefore initial rate of increase of current at $t = 0$ is

$$\frac{dI}{dt} = \frac{I_0}{\tau}$$

$$= \frac{0.416}{0.625} = 0.66 \text{ A/sec.}$$

18. Prove that the equation of continuity $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

[2019 Spring]

$\Rightarrow$ Maxwell's fourth equation is

$$\nabla \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right]$$

Taking divergence on both sides,

$$\nabla \cdot \nabla \times \vec{B} = \nabla \cdot \mu_0 \left[\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right]$$

$$\text{or, } 0 = \nabla \cdot \vec{J} + \epsilon_0 \frac{\partial (\nabla \cdot \vec{E})}{\partial t} \quad \text{since divergence of curl is zero.}$$

Now, using Maxwell's first equation in differential form,

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \text{ in above equation.}$$

$$\text{Hence } \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \text{ proved.}$$

19. A light beam travelling in x direction is described by electric field $E_y = 300 \sin \omega(t-x/c) \text{ V/m}$. An electron is constrained to move in y direction with speed $2 \times 10^7 \text{ m/s}$. Find maximum electric and magnetic force on electron. [2021 Fall]

$$\Rightarrow \text{Maximum electric field } (E_0) = 300 \text{ V/m}$$

$$\text{Maximum magnetic field } (B_0) = \frac{E_0}{c}$$

$$= \frac{300}{3 \times 10^8} = 1 \mu\text{T}$$

$$\text{Maximum electric force } (F_E) = eE_0$$

$$= 1.6 \times 10^{-19} \times 300$$

$$= 4.8 \times 10^{-16} \text{ N}$$

$$\text{Maximum magnetic force } (F_B) = B_0 e v$$

$$= 10^{-6} \times 1.6 \times 10^{-19} \times 2 \times 10^7$$

$$= 3.2 \times 10^{-18} \text{ N}$$

EXERCISE

1. A damped LCR oscillation with 100Ω resistance is given by the differential equation

$$\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 5q = 0.$$

Find:

- capacitance
- inductance
- relaxation time
- undamped frequency
- damped frequency
- quality factor.

Ans: 8mF, 25H, 0.25s, 0.35 Hz, 0.16 Hz, 0.25

2. An LR circuit containing $L = 10\text{H}$, $R = 2\Omega$ with source voltage 10V . Find the time to reach magnetic energy $\frac{1}{4}$ of its initial value.

Ans: 3.47 s

3. Derive a differential equation of LC oscillation. Show that the frequency of such oscillation is $f = \frac{30}{\pi\sqrt{LC}}$ cycles/min.

4. In a damped LC circuit, find the time required for the maximum energy present in the capacitor during one oscillation to fall to one half of its initial value in terms of τ . Assume that $q = q_0$ at $t = 0$.

Ans: 0.693\tau

5. What is the average magnitude of the Poynting vector 5 miles from a radio transmitter broadcasting isotropically with an average power of 250kW ? Calculate the amplitude of electric and magnetic field.

Ans: 307\mu W/m^2, 0.48V/m, 1.6 \times 10^{-9} T

6. Calculate the value of Poynting vector for a 60W bulb at a distance of 0.5m from it.

Ans: 19.1W/m^2

7. Prove that E and B are perpendicular to each other in an electromagnetic wave.
8. What are the peak values of electric and magnetic field vectors if the intensity of electromagnetic wave is 0.03 W/m^2 ?
Ans: 4.756 Vm^{-1} , 15.85 nT
9. Sunlight strikes the earth outside the atmosphere with an intensity of 2 cal/cm^2 per min. Calculate the E_m and B_m for sunlight.
Ans: 1026.4 V/m , $1.71 \mu\text{T}$
10. Using Maxwell's equations, prove the charge conservation theorem $\frac{d\rho}{dt} + \nabla \cdot \vec{J} = 0$.

CHAPTER 7

QUANTUM MECHANICS

- 7.1 Introduction
 - 7.1.1 Physical Significance of Wave Function Ψ
- 7.2 Schrödinger's Wave Equation
- 7.3 Potential Barrier
- 7.4 Potential Well
- 7.5 Normalize Wave Function

7.1 Introduction

According to de Broglie, a moving particle, whatever its nature, has wave properties associated with it. He proposed that the wavelength λ associated with any moving particle of momentum p is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

where h is Planck's constant. Such waves associated with the matter particle are called *matter waves*. Further the Heisenberg's uncertainty principle states that "it is impossible to determine the precisely and simultaneously the values of momentum and position of atomic particles". Therefore, the particle exhibits the wave behaviour.

According to Heisenberg, if Δx be the error in determining position and Δp is the error in determining momentum at the same instant, then these quantities are related as follows:

$$\Delta x \Delta p = h$$

where

$$\hbar = \frac{h}{2\pi} \text{ is called reduced Planck's constant.}$$

Inadequacy of Classical Mechanics

Classical mechanics successfully describe the everyday activities such as motion of bodies, planets, etc, which is mainly based on Newton's three laws of motion. Classical mechanics fails to explain the phenomenon in atomic or molecular dimensions such as motion of electrons, atomic spectra etc. which can be

explains by quantum mechanics successfully. Some inadequacies of classical mechanics are presented below:

1. Classical mechanics could not explain the stability of the atoms.
2. It does not explain the black body radiation
3. It could not explain the observed variation of specific heat of solid.
4. It could not explain the origin of the discrete spectrum of the atoms.
5. It could not explain many other atomic phenomena such as Compton effect, photoelectric effect, Raman effect, etc.

These inadequacies of classical mechanics led to the development of quantum mechanics.

Energy and Momentum Operator

If we take a physical state and do something to it, like differentiating, or rotating, we will get different state, and then we say that performing an operation on a state produces a new state. We can express this idea by an equation

$$\hat{A} \Psi = \Phi$$

An operation on state Ψ produces another state Φ . An operator A has main role in the operation. In general, an operator is defined as a mathematical term such that its operation on any state produces an other state. Addition, subtraction, multiplication etc are examples of algebraic operators. In quantum mechanics, we should operate the given function to find the respective physical quantities. Some physical quantities such as energy and momentum can be obtained by operating the given function by their respective operator.

i. Energy operator ($\hat{E}$) = $i\hbar \frac{\partial}{\partial t}$

ii. Momentum operator ($\hat{P}_x$) = $-i\hbar \frac{\partial}{\partial x}$

Momentum operator in three dimension is

$$\hat{P} = -i\hbar \nabla$$

Where $\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$ is gradient operator.

7.1.1 Physical Significance of Wave Function Ψ

In classical mechanics, the wave is represented by equation

$$\Psi = Ae^{\frac{-i}{\hbar}(Et - px)} \quad \dots\dots\dots (1)$$

This function in general is a complex quantity and is dependent upon space and time and is denoted by

$$\Psi(x, t) = Ae^{\frac{-i}{\hbar}(Et - px)} \quad \dots\dots\dots (2)$$

This function has no physical significance itself. The only quantity having physical meaning is the square of its magnitude.

$$\text{i.e., } P = \Psi^* \Psi = |\Psi|^2$$

This is called *probability density*, where Ψ^* is complex conjugate of Ψ . The probability of finding particle in the volume element dv can be expressed as $|\Psi|^2 dv$ and the probability of finding particle in entire space is $\int_{-\infty}^{\infty} |\Psi|^2 dv = 1$.

7.2 Schrödinger's Wave Equation

The quantity that characterizes the de Broglie wave is called *wave function*. It is denoted by ψ .

The wave function of a particle in one dimension is

$$\Psi(x, t) = Ae^{\frac{-i}{\hbar}(Et - px)} \quad \dots\dots\dots (i)$$

Differentiating equation (i) with respect to x ,

$$\frac{d\Psi}{dx} = \frac{ip}{\hbar} Ae^{\frac{-i}{\hbar}(Et - px)} \quad \dots\dots\dots (ii)$$

Again differentiating with respect to x ,

$$\begin{aligned} \frac{d^2\Psi}{dx^2} &= \left(\frac{ip}{\hbar}\right)^2 Ae^{\frac{-i}{\hbar}(Et - px)} \\ &= \frac{-p^2}{\hbar^2} \Psi \quad \dots\dots\dots (iii) \end{aligned}$$

Total energy of a particle is

$$\begin{aligned} E &= K + V \\ &= \frac{1}{2}mv^2 + V = \frac{p^2}{2m} + V \end{aligned}$$

Multiplying both sides by Ψ and rearranging, we get

$$p^2\Psi = 2m(E - V)\Psi \quad \dots\dots\dots (iv)$$

Then, equation (iii) becomes

$$\begin{aligned} \frac{d^2\Psi}{dx^2} &= \frac{-2m(E - V)}{\hbar^2} \Psi \\ \therefore \frac{d^2\Psi}{dx^2} + \frac{2m(E - V)}{\hbar^2} \Psi &= 0 \quad \dots\dots\dots (v) \end{aligned}$$

This is the time independent Schrodinger's wave equation in one dimension.

Three-dimensional time independent -Schrodinger's wave equation is

$$\nabla^2 \Psi + \frac{2m(E - V)}{\hbar^2} \Psi = 0$$

where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ is an operator known as

Laplacian operator.

Differentiating equation (2) with respect to time, we get

$$\frac{d\Psi}{dt} = A \left(\frac{-i}{\hbar} \right) E e^{\frac{-i}{\hbar}(Et - px)} = -i \frac{E}{\hbar} \Psi$$

$$\text{or, } i\hbar \frac{d\Psi}{dt} = E \Psi \dots\dots\dots(3)$$

Differentiating $\Psi(x, t)$ w.r. to x , we have

$$\frac{d\Psi}{dx} = A \frac{ip}{\hbar} e^{\frac{-i}{\hbar}(Et - px)} \dots\dots\dots(4)$$

Differentiating again will result

$$\frac{d^2\Psi}{dx^2} = A \left(\frac{-i}{\hbar} \right)^2 e^{\frac{-i}{\hbar}(Et - px)}$$

$$\text{or, } \frac{d^2\Psi}{dx^2} = -\frac{p^2}{\hbar^2} \Psi \dots\dots\dots(5)$$

Total energy, $E = K + V$

$$= \frac{p^2}{2m} + V$$

$$\text{or, } p^2 = 2m(E - V)$$

Operating this operator on Ψ , we get

$$p^2 \Psi = 2m(E - V)\Psi$$

From equation (5),

$$-\hbar^2 \frac{d^2\Psi}{dx^2} = 2m(E - V)\Psi \dots\dots\dots(6)$$

Using equation (3) in (6), we get

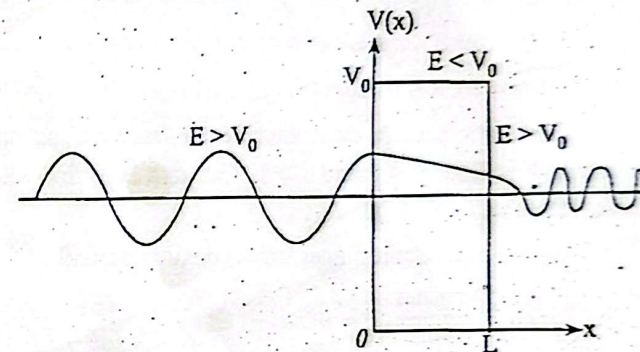
$$-\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} + V \Psi = i\hbar \frac{d\Psi}{dt}$$

In three dimension,

$$i\hbar \frac{d\Psi}{dt} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi$$

This is the required Schrödinger time dependent wave equation.

7.3 Potential Barrier



Let's consider a finite potential $V(x)$ energy function of height V_0 in the region of width L . The downward facing region of the potential function is called *potential barrier*. For a particle, it is barrier because a particle should have minimum energy of V_0 in order to reach the other side. However, if the barrier is relatively narrow, the particle can tunnel through to cross the barrier. Transmission probability of particle is given by

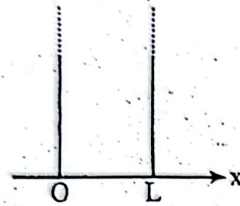
$$T = \frac{16E}{V} \left(1 - \frac{E}{V} \right) e^{-2\alpha L}$$

where E is energy of particle, V is barrier potential, L is length of the barrier, α is a constant which is given by

$$\alpha = \sqrt{\frac{2m(V - E)}{\hbar^2}}, \text{ and } m \text{ is the mass of the incident particle.}$$

7.4 Potential Well

Consider a particle moving inside box along x-direction. The box has infinite potential barriers at $x = 0$ and at $x = L$.



Potential function is

$$V = 0 \text{ for } 0 < x < L$$

$$= \infty \text{ for } x < 0 \text{ and } x > L \dots\dots(1)$$

The particle can't exist outside the box and so its wave function Ψ is 0 for $x < 0$ and $x > L$. Our task is to find what Ψ is within the box.

Within box, Schrödinger wave equation becomes

$$\frac{d^2\Psi}{dx^2} + \frac{2m(E-V)}{\hbar^2} \Psi = 0$$

$$\text{or, } \frac{d^2\Psi}{dx^2} + \frac{2mE}{\hbar^2} = 0 \dots\dots(2) \quad \because V = 0 \text{ within box}$$

$$\text{Let } \frac{2mE}{\hbar^2} = k^2, \text{ then}$$

$$\frac{d^2\Psi}{dx^2} + k^2\Psi = 0$$

General solution of this equation is

$$\Psi = A \sin kx + B \cos kx \dots\dots(3)$$

The boundary conditions can be used to evaluate the constants A and B in equation (3).

$$\Psi = 0 \text{ at } x = 0 \Rightarrow B = 0$$

$$\Psi = 0 \text{ at } x = L \Rightarrow A \sin kL = 0$$

$$\text{Since } A \neq 0, \sin kL = 0$$

$$\text{or, } kL = n\pi \dots\dots(4)$$

This will make equation (3) reduced to

$$\Psi = A \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore \Psi_n(x) = A \sin\left(\frac{n\pi x}{L}\right)$$

Squaring equation (4), we have

$$\text{or, } k^2 L^2 = n^2 \pi^2$$

$$\text{or, } \frac{2mEL^2}{\hbar^2} = n^2 \pi^2$$

$$\therefore E_n = \frac{n^2 \pi^2 \hbar^2}{2m L^2}$$

For each n, there is an energy level and corresponding wave function. Each value of E_n is called *Eigen value* and corresponding Ψ_n is called *Eigen function*.

Since $\hbar$, m , π , and L are fixed and value of n gives only discrete level of energy, we can conclude that the particle can have only discrete energy.

7.5 Normalize Wave Function

Since the probability of finding the particle within the well is certain

$$\text{i.e. } \int_0^L |\Psi|^2 dx = 1. \quad \text{Where, } \Psi = A \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{or, } \int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = 1.$$

$$\text{or, } \int_0^L A^2 \frac{1}{2} \left[1 - \cos\left(\frac{2n\pi x}{L}\right) \right] dx = 1$$

$$\text{or, } \frac{A^2}{2} \left[\int_0^L dx - \int_0^L \cos\left(\frac{2n\pi x}{L}\right) dx \right] = 1$$

$$\text{or, } \frac{A^2}{2} (L - 0) = 1$$

$$\therefore A = \sqrt{\frac{2}{L}}$$

Hence the normalized wave function is $\Psi = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

SOLUTION TO EXAM PROBLEMS AND IMPORTANT QUESTIONS

1. Calculate the permitted energy levels of an electron in one dimensional potential well of width 0.2 nm.

$$\Rightarrow \text{Width of potential well } (l) = 0.2 \text{ nm} \\ = 2 \times 10^{-10} \text{ m}$$

Permitted energy level of an electron in one dimensional potential well is given by

$$E_n = \frac{\pi^2 \hbar^2}{2m l^2} n^2 \\ = \frac{\pi^2 (1.05 \times 10^{-34})^2}{2 \times 9.1 \times 10^{-31} \times (2 \times 10^{-10})^2} n^2 \\ = 1.5 \times 10^{-18} n^2 \text{ J}$$

In eV,

$$E_n = \frac{1.5 \times 10^{-18}}{1.6 \times 10^{-19}} n^2 \text{ eV} = 9.38 n^2 \text{ eV}$$

Minimum energy that electron have is $E_1 = 9.38 \text{ eV}$ corresponding to $n = 1$.

Other values of energies are

$$E_2 = 4E_1 \\ = 37.5 \text{ eV}$$

$$E_3 = 9E_1 \\ = 84.3 \text{ eV, and so on.}$$

2. A beam of electrons having energy of each 3 eV is incident on a potential barrier of height 4 eV. If the width of the barrier is 20 Å, calculate the transmission coefficient of the beam through the barrier.

$$\Rightarrow \text{Energy of electron } (E) = 3 \text{ eV} \\ \text{Barrier height } (V) = 4 \text{ eV}$$

$$\begin{aligned}\text{Barrier width (L)} &= 20 \text{ \AA} \\ &= 20 \times 10^{-10} \text{ m}\end{aligned}$$

Transmission coefficient (T) = ?

We have,

$$\begin{aligned}T &= 16 \frac{E}{V} \left(1 - \frac{E}{V}\right) \exp \left[-\frac{2L}{h} \sqrt{2m(V-E)} \right] \\ &= 16 \left(\frac{3}{4}\right) \left(1 - \frac{3}{4}\right) \\ &\quad \exp \left[-\frac{2 \times 20 \times 10^{-10}}{1.05 \times 10^{-34}} \times \sqrt{2(9.1 \times 10^{-31}) \cdot (4-3) \times 1.6 \times 10^{-19}} \right] \\ &= 3.7 \times 10^{-9}\end{aligned}$$

Since the transmission coefficient is very low, probability of transmission is very low.

3. An electron is confined to an infinite height box of size 0.1 nm. Calculate the ground state energy of the electron. How this electron can be put to the third energy level?

$$\Rightarrow \text{Mass of the electron (m)} = 9.1 \times 10^{-31} \text{ kg}$$

$$L = 0.1 \text{ nm} = 0.1 \times 10^{-9} \text{ m}$$

$$E_n = ?$$

$$\begin{aligned}\text{The permitted electron energies} &= E_n = \frac{n^2 h^2}{8mL^2} \\ &= n^2 \frac{(6.626 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (0.1 \times 10^{-9})^2 \times 1.6 \times 10^{-19}} \\ &= 38n^2 \text{ eV}\end{aligned}$$

$$\text{Ground state energy is } E_1 = 38 \text{ eV}$$

$$\begin{aligned}\text{Energy of electron in 3rd level is } E_3 &= 9E_1 \\ &= 342 \text{ eV}\end{aligned}$$

$$\begin{aligned}\text{Difference in energy is } E_3 - E_1 &= (342 - 38) \text{ eV} \\ &= 304 \text{ eV}\end{aligned}$$

$$\text{As } E = \frac{hc}{\lambda}$$

$$\text{or, } \lambda = \frac{hc}{E}$$

$$\begin{aligned}&= \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{304 \times 1.6 \times 10^{-19}} \\ &= 4.08 \times 10^{-9} \text{ m}\end{aligned}$$

By striking electron with photon of wavelength 408 nm, electron can be put into third energy level.

4. A football of 500 gm is confined between two impenetrable walls of Dasharath Rangashala that can be treated as a box of length 100m. Calculate the minimum speed of the ball.

$$\Rightarrow \text{Mass of football (m)} = 500 \text{ gm} = 0.5 \text{ kg}$$

$$\text{Width of box (L)} = 100 \text{ m}$$

$$\text{Minimum speed (v}_{\min}) = ?$$

We have,

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

For minimum energy, $n = 1$

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2}$$

$$\text{or, } \frac{1}{2} m v_{\min}^2 = \frac{\pi^2 \hbar^2}{2mL^2}$$

$$\text{or, } v_{\min} = \frac{\pi \hbar}{mL}$$

$$\begin{aligned}\therefore v_{\min} &= \frac{3.1415 \times 1.05 \times 10^{-34}}{0.5 \times 100} \\ &= 6.62 \times 10^{-36} \text{ ms}^{-1}\end{aligned}$$

5. A non relativistic particle is moving three times as fast as an electron. The ratio of the de Broglie wavelength of the particle to that of the electron is 1.813×10^{-4} . Calculate the mass of the particle.

$\Rightarrow$ Let the velocity of particle and an electron be v_1 and v_2

From question, $v_1 = 3v_2$(i)

Now, de Broglie wavelength of the particle and an electron be

$$\lambda_1 = \frac{h}{m_1 v_1} \text{(ii)}$$

$$\lambda_2 = \frac{h}{m_2 v_2} \text{(iii)}$$

From equations (ii) and (iii),

$$\frac{\lambda_1}{\lambda_2} = \frac{m_2 v_2}{m_1 v_1}$$

$$= 1.813 \times 10^{-4}$$

$$\frac{m_2 v_2}{m_1 3v_2} = 1.813 \times 10^{-4}$$

$$\text{Mass of particle } (m_1) = \frac{m_2}{3 \times 1.813 \times 10^{-4}}$$

$$m_1 = \frac{9.1 \times 10^{-31}}{3 \times 1.813 \times 10^{-4}}$$

$$= 1.673 \times 10^{-27} \text{ kg}$$

6. An electron is confined to a infinite potential well of size 8.5 nm . Calculate the ground state energy of an electron and radian frequency. Given: Plank's constant $= 6.62 \times 10^{-34} \text{ Js}$, mass of electron $= 9.1 \times 10^{-31} \text{ kg}$.

$\Rightarrow$ We know energy of an electron in infinite potential well is

$$E = \frac{n^2 h^2}{8mL^2}, \quad n=1 \text{ for ground state.}$$

$$= \frac{(6.62 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} (8.5 \times 10^{-9})^2}$$

$$= 8.88210^{-22} \text{ Joule}$$

Again, frequency of radiation is $E = hf$

$$\text{Or, } f = \frac{E}{h} = 12.66 \times 10^{12} \text{ Hz.}$$

7. An electron moving as a wave has wave function $\psi(x) = 2\sin 2\pi x$. Find the probability of the electron forming in the region $x = 0.25$ to 0.5 m . [2014 Spring]

$\Rightarrow$ Probability of finding electron in the given region is

$$P = \int_{0.25}^{0.5} |\Psi|^2 dx = \int_{0.25}^{0.5} 4 \sin^2 2\pi x dx$$

$$= 4 \int_{0.25}^{0.5} \frac{1}{2} (1 - \cos 4\pi x) dx$$

$$= 2 [x]_{0.25}^{0.5} - 2 \left(\frac{\sin 4\pi x}{4\pi} \right)_{0.25}^{0.5}$$

$$= 0.5$$

Hence the probability of finding electron in the given region is 50%

8. Normalize the one dimensional wave function

$$\Psi = A \sin\left(\frac{n\pi x}{a}\right), \quad 0 < x < a$$

$$\Psi = 0, \quad \text{outside}$$

[2014 Fall]

$\Rightarrow$ For normalized wave function,

$$\text{i.e. } \int_0^a |\Psi|^2 dx = 1. \quad \text{Where, } \Psi = A \sin\left(\frac{n\pi x}{a}\right)$$

$$\text{or, } \int_0^a A^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = 1.$$

$$\text{or, } \int_0^a A^2 \frac{1}{2} \left[1 - \cos\left(\frac{2nx}{a}\right) \right] dx = 1$$

$$\text{or, } \frac{A^2}{2} \left[\int_0^a dx - \int_0^a \cos\left(\frac{2nx}{a}\right) dx \right] = 1$$

$$\text{or, } \frac{A^2}{2} (a - 0) = 1$$

$$\therefore A = \sqrt{\frac{2}{a}}$$

Hence the normalized wave function is

$$\Psi = \sqrt{\frac{2}{a}} \sin\left(\frac{nx}{a}\right)$$

9. The fast moving neutron has wave associated with it, whose De-Broglie wavelength is $2 \times 10^{-12} \text{ m}$. Find phase velocity, group velocity and kinetic energy. [2015 Spring]

$$\Rightarrow \text{The De-Broglie wavelength is } (\lambda) = \frac{h}{mv}$$

$$\text{or, } v = \frac{h}{m\lambda}$$

$$= \frac{6.02 \times 10^{-34}}{6.67 \times 10^{-27} \times 2 \times 10^{-12}}$$

$$\therefore v = 4.5 \times 10^4 \text{ m/s}$$

For matter wave both the group velocity and the phase velocity are equal.

Now, kinetic energy of the neutron is

$$E = \frac{1}{2} mv^2$$

$$= \frac{1}{2} \times 6.67 \times 10^{-27} (4.5 \times 10^4)^2$$

$$= 6.75 \times 10^{-18} \text{ Joule.}$$

10. Calculate the de-Broglie wavelength produced by applying 15kV energy. (Take mass of proton = $1.67 \times 10^{-27} \text{ kg}$). [2022 Fall]

$$\Rightarrow \text{The De-Broglie wavelength is } (\lambda) = \frac{h}{mv}$$

For the particle with applying potential V,

$$eV = \frac{1}{2} mv^2$$

$$\text{or, } v = \sqrt{\frac{2eV}{m}}$$

So de-Broglie wavelength :

$$(\lambda) = \frac{h}{\sqrt{2meV}}$$

$$= \frac{6.02 \times 10^{-34}}{\sqrt{2 \times 6.67 \times 10^{-27} \times 1.6 \times 10^{-19} \times 15000}}$$

$$\therefore \lambda = 6.393 \times 10^{-15} \text{ m}$$

Which shows that the de-Broglie wavelength for matter is very small.

◆◆◆

EXERCISE

1. An electron is confined to an infinite height box of size 0.1 nm. Calculate the ground state energy of the electron. How much energy can be put to the third energy level?
Ans: 37eV, $\lambda_{\text{photon}} = 4.12\text{nm}$
2. Find the minimum energy of an electron in a box of width 1.5Å.
Ans: 16.14 eV
3. Using uncertainty principle, calculate the binding energy of a hydrogen atom.
Ans: -13.6eV
4. Calculate the probability of transmission that a 1eV electron will penetrate a potential barrier of 4.0eV when the barrier width is 2.0Å.
Ans: 0.048
5. An α -particle of mass 7000 times of an electron is trapped in a nucleus whose radius is $r_0 = 1.4\text{fm}$. What is the probability that an α -particle will escape from the nucleus if its energy is 2MeV? The potential barrier at the surface of the nucleus is 4MeV.
Ans: 0.124
6. Calculate the permitted energy levels of an electron in one dimensional potential well of width 0.2nm.
Ans: $1.5 \times 10^{-8} n^2 \text{joule}$

CHAPTER 8

FUNDAMENTALS OF THERMODYNAMICS AND HEAT TRANSFER

8.1 Introduction

8.1.1 Thermodynamic Properties

8.1.2 Thermodynamic State and Equilibrium

8.1.3 Thermodynamic Process

8.2 First Law of Thermodynamics

8.2.1 Application of First Law of Thermodynamics for Control Mass

8.2.2 Second Law of Thermodynamics

8.2.3 Heat Engine

8.2.4 Heat Pump

8.2.5 Refrigerator

8.3 Heat Transfer

8.3.1 Conduction

8.3.2 Convection

8.3.3 Radiation

Chapter	Fundamentals of
8	Thermodynamics and Heat Transfer

8.1 Introduction

The term 'thermodynamics' consist of 'thermo' meaning heat and 'dynamics' meaning motion or movement. Hence, thermodynamics is the branch of physics which deals with work done and heat energy.

Today the same name is broadly interpreted to include all aspects of energy and energy transformations, including power generation, refrigeration, and relationships among the properties of matter. Thus, basic engineering thermodynamics is concerned with the study of:

- Interaction of system and surroundings which relates the changes in the system due to the interaction in the form of mass and energy transfer (heat and work transfer).
- Energy and its conversion from one form to another to produce certain useful effects.
- Fundamental thermodynamic laws which govern the energy conversion processes.
- Physical properties (pressure, temperature and volume) of the working substances which can be used to obtain energy conversion.
- The concept of an equilibrium condition and feasibility of a process.

8.1.1 Thermodynamic Properties

Thermodynamic properties is defined as variables or characteristics which can be used to describe the state or condition of a system, for example mass, temperature, pressure, volume,

energy, entropy, enthalpy, etc. According to their nature, thermodynamic properties can be divided in to two groups:

a. Intensive properties

The properties which are independent to the mass of the system are called *intensive properties*. Its value remains the same whether one considers the whole system or only a part of it. Intensive property is not additive by nature. Examples are pressure, temperature, density, velocity, viscosity, thermal conductivity, electrical potentials, etc.

b. Extensive properties

The properties which depend on the size or extent of the system are called *extensive properties*. Its value depends on how big a portion of the system is being considered. Extensive property is additive by nature. Examples are mass, volume, energy, enthalpy, entropy, surface area, etc.

8.1.2 Thermodynamic State and Equilibrium

A system is said to exist in a condition of *thermodynamic equilibrium* if there is no change in any macroscopic property of the system when it is isolated from its surroundings. The condition of a thermodynamic system can be described with the help of the values of different thermodynamic properties, e.g. temperature, pressure, volume etc. To describe a system with the values of certain thermodynamic properties it should have same values of those properties throughout the system. Therefore, there can be no spontaneous change in any macroscopic properties for the system to be in a condition of thermodynamic equilibrium. Hence, thermodynamics is a study of mainly the properties of physical systems that are found in equilibrium condition. A thermodynamic system will be in a condition of thermodynamic equilibrium, if it satisfies the following three types of equilibrium conditions

a. Thermal equilibrium

A system is said to be in *thermal equilibrium* if the temperature is same throughout the system.

b. **Mechanical equilibrium**

A system is said to be in *mechanical equilibrium* if there is no change in pressure with time.

c. **Chemical equilibrium**

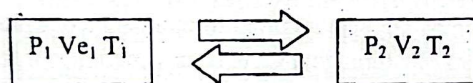
A system is said to be in *chemical equilibrium* if its chemical composition does not change with time.

8.1.3 Thermodynamic Process

Any change in thermodynamic variables of a system to have new set of values is called *process*. *Thermodynamic process* is the path followed by a system when it undergoes from one equilibrium state to another. There are two types of thermodynamic process:

a. **Reversible process**

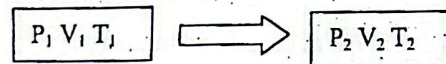
A process is said to be *reversible* if the system can be retraced through the same path to the initial state.



Examples: Frictionless motion, elastic deformation, restricted expansion and compression, etc.

b. **Irreversible process**

A process is said to be *irreversible* if the system cannot regain its original state through the initial path.



Examples: Motion with friction, inelastic deformation, free expansion, etc.

8.2 First Law of Thermodynamics

First law of thermodynamics is also known as the principle of conservation of energy. If dQ amount of heat is supplied to a system, a part of heat (dU) is used to increase the internal energy and remaining part of heat (dW) is used to do the external work. If dQ , dU , and dW all are in same unit we can write

$$dQ = dU + dW$$

$$\text{or, } dQ = dU + PdV$$

which are the mathematical expression of the first law of thermodynamics.

First law of thermodynamics is based on the conservation principles namely mass and energy conservation principles. It is a mathematical expression for the effect of the interactions between the system and surrounding on the stored energy or total energy (internal energy, kinetic energy and potential energy) of the system. In closed system or control mass, interactions can take place only in the form of energy transfer (e.g., work transfer and heat transfer) but no mass transfer occurs between the system and surroundings. Therefore, first law of thermodynamics for a control mass can be explained with reference to conservation of energy only whereas in an open system or control volume, interactions can take place both by the mass transfer and energy transfer. Hence, the first law of thermodynamic for a control volume is explained with reference to both mass conservation and energy conservation principles.

8.2.1 Application of First Law of Thermodynamics for Control Mass

I. Isothermal Process (Constant Temperature)

It is a thermodynamical process in which the temperature of the system remains constant but other parameters changes.

From an ideal gas equation,

$$PV = nRT$$

But T is constant in isothermal process so nR is also constant and the equation can be written as

$$P_1 V_1 = P_2 V_2 = P_3 V_3 = \text{Constant}$$

Let dW be the small work done on changing small volume dV

Then we have, $dW = PdV$

$$\begin{aligned}\text{Total work done is } W &= \int_{V_1}^{V_2} P dV \\ &= \int_{V_1}^{V_2} \frac{nRT}{V} dV\end{aligned}$$

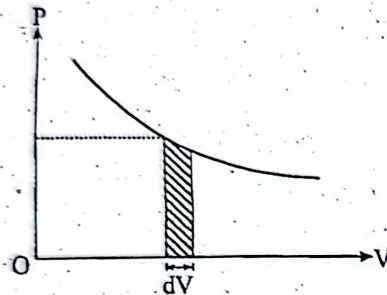


Fig.: Isothermal P-V curve

$$W = nRT \ln\left(\frac{V_2}{V_1}\right)$$

which shows that heat transfer to a control mass for an ideal gas during isothermal process is equal to the work done by the system.

II. Isochoric Process (Constant Volume)

It is a thermodynamical process in which the volume of the system remains constant but other parameters changes.

From an ideal gas equation,

$$\text{For isochoric process, } \frac{P}{T} = \text{constant}$$

$$\text{Total work done } (W) = \int_{V_1}^{V_2} P dV = 0$$

So from first law of thermodynamics;

$$dQ = dU$$

which shows that heat transfer to a control mass for an ideal gas during isochoric process is equal to the increase in internal energy of the system.

III. Isobaric Process (Constant Pressure)

It is a thermodynamical process in which the pressure of the system remains constant but other parameters changes.

From an ideal gas equation,

$$\frac{V}{T} = \text{constant}$$

$$\text{Total work done } (W) = \int_{V_1}^{V_2} P dV = P(V_2 - V_1)$$

Which shows that heat transfer to a control mass for an ideal gas during isobaric process is equal to both the increase in internal energy and the external work done of the system; which means increase in enthalpy of the system.

IV. Specific Heat Capacity

Specific heat capacity is defined as the amount of heat required to raise the temperature of unit mass of a substance through 1°C . For an ideal gas, we define molar heat capacity instead of specific heat capacity. *Molar heat capacity* is defined as the heat required to raise the temperature of unit mole of gas through 1°C . It is an experimental fact that for same increase in temperature heat supplied is different for different way of heating up the system. Correspondingly the heat capacity depends on the manner of heating. Commonly considered heat capacities are C_V and C_P , which respectively correspond to heating at constants V and P . Heat capacities per unit mass per mole of a substance are called specific heats.

a. Specific heat at constant volume (C_V)

It is defined as the heat required to raise the temperature of unit mole of gas through 1°C at constant volume.

$$\text{i.e. } dQ = dU + PdV$$

$$= nC_V dT$$

$$\text{or, } C_V = \left(\frac{dQ}{dT}\right)_V$$

b. Specific heat at constant pressure (C_p)

It is defined as the heat required to raise the temperature of unit mole of gas through 1°C at constant pressure.

$$\text{or, } C_p = \left(\frac{dQ}{dT} \right)_p$$

V. Polytropic Process

A *polytropic process* is a thermodynamic process that obeys the pressure - volume relation: $PV^n = \text{constant}$, where n is any real number called a polytropic index or exponent whose value remains constant throughout in any particular process. This equation is used to accurately characterize the thermodynamic processes of certain systems notably the compression or expansion of a gas. However, it can be taken as a generalized equation for different thermodynamic processes namely isochoric, isobaric, isothermal and adiabatic processes which are quasi-static or quasi-equilibrium in operation under ideal working conditions. A polytropic process represents common thermodynamic processes for different values of polytropic index n as given below:

S.N.	Polytropic index (n)	Relation	Process
1	0	$P = \text{constant}$	Isobaric
2	1	$PV = \text{constant}$	Isothermal
3	∞	$V = \text{constant}$	Isochoric
4	γ	$PV^\gamma = \text{constant}$	Adiabatic

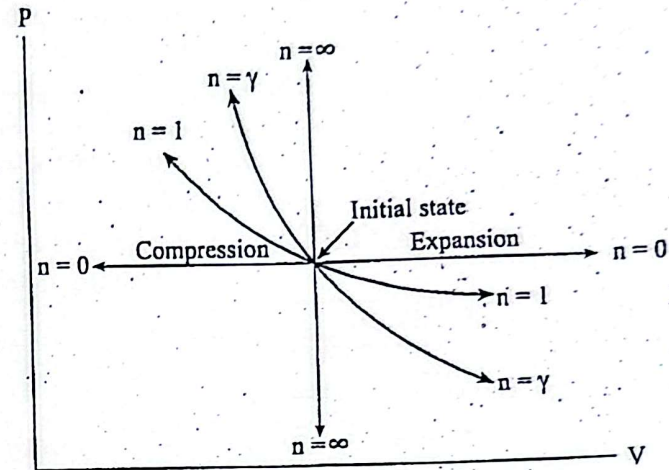


Fig.: Polytropic process

Work done in polytropic process can be obtained by integrating small work done.

$$dW = PdV \dots\dots\dots(i)$$

$$\text{For polytropic process } P_1V_1^n = P_2V_2^n \\ = PV^n$$

$$\text{or, } P = \frac{P_1V_1^n}{V^n}$$

$$\begin{aligned} \text{Now, the total workdone is } W &= \int_{V_1}^{V_2} PdV \\ &= \int_{V_1}^{V_2} \frac{P_1V_1^n}{V^n} dV \\ &= P_1V_1^n \int_{V_1}^{V_2} \frac{dV}{V^n} \\ &= P_1V_1^n \int_{V_1}^{V_2} V^{1-n} dV \\ &= P_1V_1^n \left[\frac{V^{1-n}}{1-n} \right]_{V_1}^{V_2} \end{aligned}$$

$$= \frac{P_1 V_1^n}{1-n} (V_2^{1-n} - V_1^{1-n})$$

$$= \frac{1}{1-n} (P_1 V_1^n V_2^{1-n} - P_1 V_1)$$

Again using relation,

$$P_1 V_1^n = P_2 V_2^n$$

Required work done is

$$W = \frac{P_2 V_2 - P_1 V_1}{1-n}$$

$$\text{or, } W = \frac{mR(T_2 - T_1)}{1-n} \dots\dots\dots(ii)$$

Here change in internal energy is

$$dU = mC_v(T_2 - T_1)$$

Now, from the first law of thermodynamics,

$$Q = U + W$$

$$= mC_v(T_2 - T_1) + \frac{mR(T_2 - T_1)}{1-n}$$

$$= m \left(C_v + \frac{R}{1-n} \right) (T_2 - T_1)$$

$$= mC_n(T_2 - T_1)$$

Where,

$$C_n = \left(C_v + \frac{R}{1-n} \right) \text{ is polytropic specific heat.}$$

Limitations of first law of thermodynamics:

1. It does not show the direction of heat transfer.
2. It does not give any information about the source and sink.
3. It does not give any information about what extent heat energy used to do the external work.

8.2.2 Second Law of Thermodynamics

First law of thermodynamics deals with conservation and conversion of energy. It stipulates that when a thermodynamic process is carried out, energy is neither created nor destroyed, but it transforms from one form into another so that the energy balance is maintained. However, the first law fails to state the conditions under which energy conversions are possible. It explains thermodynamic processes with the quantitative aspect of the energy and gives the only condition that any process can take place in either direction provided that the total energy remains constant. But this is not true, because some processes in nature proceed spontaneously in only one direction and cannot occur in opposite direction even though the reversal of process does not violate the first law i.e., energy conservation principle is satisfied. Hence, such directional feature of the thermodynamic process cannot be explained only with the first law of thermodynamics. This inadequacy of the first law to identify whether a process can take place is remedied by introducing another general principle, the *second law of thermodynamics*. The second law of thermodynamics deals with quality or nature of energy and defines the direction of the thermodynamic process in which the system can proceed.

As per the first law of thermodynamics all forms of energy are equivalent and for a cyclic process either heat is converted completely into work or work is converted into heat. However, any real device, even operating on a cycle, cannot convert heat supplied to it completely into output work. In this regard, the second law of thermodynamics declares that different forms of energy are not equivalent in their ability to produce useful output work and explains why any real engine cannot operate without heat loss, i.e. heat cannot be completely converted into work.

a. Kelvin-Planck Statement

There exist no thermodynamic transformation whose sole effect is to extract a quantity of heat from a given heat reservoir and to convert it entirely in to work.

b. Clausius Statement

There exists no thermodynamic transformation whose sole effect is to extract a quantity of heat from a colder reservoir and to deliver it to a hotter reservoir.

8.2.3 Heat Engine

Heat engines are devices that work on cyclic process to perform work. Heat pump and Refrigerator are devices used for heating and cooling purpose.

An ideal engine which converts all the heat energy in to work done is called Carnot engine.

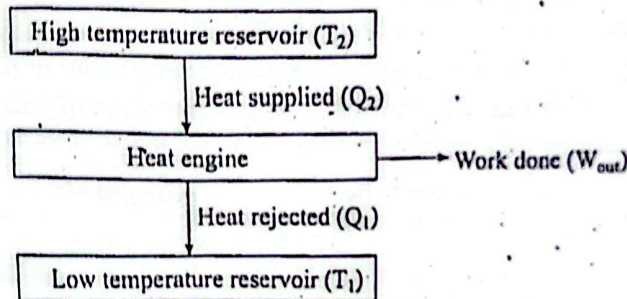


Fig.: Heat engine

The efficiency of heat engine is defined as the ratio of output work done and the heat supplied.

$$\text{i.e. efficiency}(\eta) = \frac{\text{Output work done}}{\text{Heat supplied}}$$

$$\text{or, } \eta = \frac{W}{Q}$$

$$= \frac{Q_2 - Q_1}{Q_2}$$

$$\therefore \eta = 1 - \frac{Q_1}{Q_2}$$

• In terms of temperature,

$$\text{Efficiency } (\eta) = 1 - \frac{T_1}{T_2}$$

8.2.4 Heat Pump

Heat pump also works on a cyclic process. It takes heat from low temperature reservoir (T_1) i.e. surrounding and delivers into high temperature reservoir (T_2) with the help of external work. Heat pump works as opposite to heat engine.

Efficiency of heat pump is measured by its coefficient of performance; which is defined as the ratio of output work done to the heat supplied.

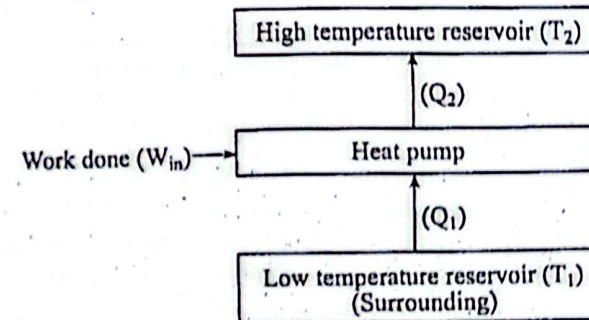


Fig.: Heat pump

$$\text{i.e. Coefficient of performance (COP)} = \frac{\text{Output work}}{\text{Input work}}$$

From the first law of thermodynamics, $W = Q_2 - Q_1$

$$\therefore \text{COP} = \frac{Q_2}{Q_2 - Q_1}$$

In terms of temperature,

$$\text{COP} = \frac{T_2}{T_2 - T_1}$$

8.2.5 Refrigerator

Refrigerator works in cyclic process which takes heat from a low temperature reservoir at temperature T_1 and delivers heat in to the high temperature reservoir at temperature T_2 with the help of external work.

Figure illustrate the schematic representation of refrigerator where Q_1 amount of heat from desired space at temperature T_1 and supplied Q_2 amount of heat to surrounding at temperature T_2 with the help of external work W .

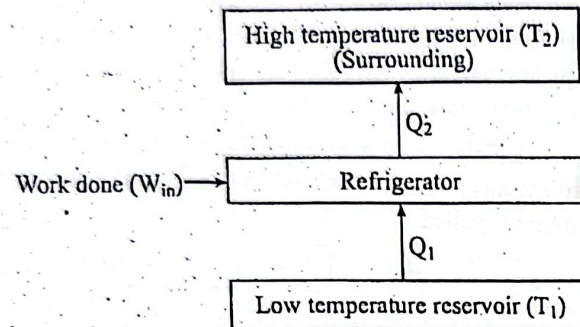


Fig.: Refrigerator

The working principle of a *refrigerator* is similar with the heat pump. It also takes heat from low temperature reservoir and delivers in to high temperature reservoir with the help of external work. The schematic diagram of refrigerator is shown in figure. First of all from the liquid storage tank working substance in the liquid form is allowed to pass through the porous plug to the evaporating tank. It suffers adiabatic expansion in passing through the plug resulting the decrease in temperature. The liquid takes latent heat from the cold body and convert in to vapor, the vapor goes to compression tank it is compressed adiabatically. Due to compression the temperature of vapor increases. Now the compressed vapor is allowed to pass in to the condenser where it lose its heat to the hot body (surrounding) and condense into liquid. Then the condense liquid again goes to the storage tank for next process.

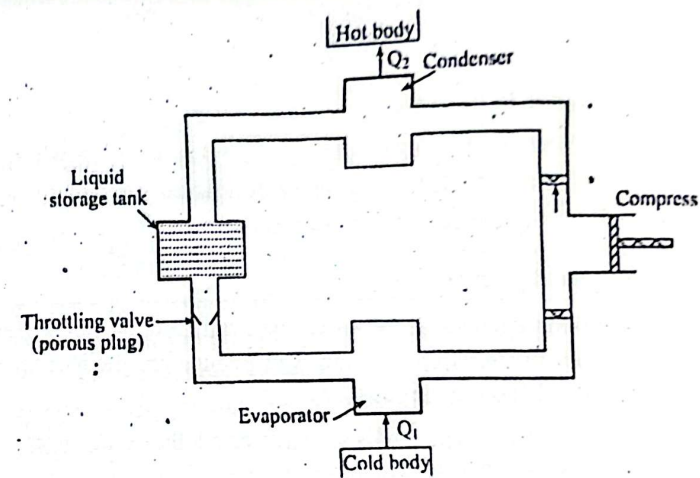


Fig.: Refrigerator

Efficiency of refrigerator is measured by its coefficient of performance (COP), which is defined as the ratio of desired effect and the work input;

$$\text{COP} = \frac{\text{Desired effect}}{\text{Input work}}$$

$$= \frac{Q_2}{Q_2 - Q_1}$$

In terms of temperature,

$$\text{COP} = \frac{T_2}{T_2 - T_1}$$

8.3 Heat Transfer

Heat transfer is generally a multidimensional and time dependent phenomenon, i.e., the temperature in a medium varies with position as well as time. For analysis of heat transfer problems, two types of heat transfer are considered, e.g. *steady state heat transfer* and *unsteady state or transient heat transfer*. Heat transfer in a medium is said to be a *steady state* if the temperature at any particular location on the system does not vary with time. The temperature is function of space coordinates only,

but it is independent of time. During steady state conditions, the heat transfer rate is constant and there is no change of internal energy of the system. In *unsteady state or transient heat transfer* the temperature varies with time as well as position or location on the system. Hence, the temperature is a function of time and space coordinates. During unsteady state heat transfer, rate of heat transfer varies with time due to change in internal energy of the system. Most of the actual heat transfer processes are unsteady in nature, but some of them are considered as steady state to simplify them.

A heat transfer analysis has applications in the design of electronic components, electric machines, transformers, bearings etc. to avoid the overheating and damage of equipment where it requires considerations of the amount of heat to be transmitted as well as the rate at which heat is to be transferred.

8.3.1 Conduction

Conduction is the transfer of heat from one part of a substance to another part of the same substance or from one substance to another in physical contact with it without appreciable movement of molecules or atoms forming the substance. In fact, heat conduction is due to the property of matter which allows the passage for heat energy even its parts are not in motion relative to one another. The heat conduction occurs more effectively in solids than in liquids and gases.

Fourier Law of Thermal Conductivity

Magnitude of conduction heat transfer is given by Fourier's law of heat conduction which is an empirical law based on observation and states as:

The rate of flow of heat per unit area (heat flux) through a homogeneous solid is directly proportional to the temperature gradient within the solid material.

Mathematically, it can be represented by the equation as

$$\frac{1}{A} \frac{dQ}{dT} \propto \frac{dT}{dx}$$

$$\text{or, } \frac{1}{A} \frac{dQ}{dT} = -k \frac{dT}{dx}$$

Negative sign indicates that heat flows in the direction of decreasing temperature.

k = a proportionality constant known as thermal conductivity of material (watt/mK).

$\frac{dT}{dx}$ = temperature gradient within the material.

$$\therefore \frac{dQ}{dT} = -kA \frac{dT}{dx}$$

One Dimensional Steady State Heat Conduction through Plane Wall

Let us consider steady heat conduction through the plane walls of a building. Heat is continuously lost to the surroundings through the wall during a winter day.

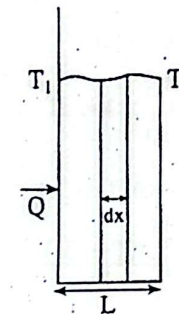


Fig.: One dimensional heat conduction through a plane wall

It can be observed that heat transfer through the wall is in the perpendicular direction to the wall surface, and no significant heat transfer takes place in the wall in other directions as shown in figure. Heat transfer in a certain direction is driven by the

temperature gradient in that direction. There is no heat transfer in a direction in which there is no change in temperature. It can be confirmed from the temperature measurements that the temperatures at the top and bottom of a wall surface as well as at the right and left ends are almost the same. Therefore, there is no heat transfer through the wall from the top to the bottom or from left to right, but there is considerable temperature difference between the inner and the outer surfaces of the wall, and thus significant heat transfer in the direction from the inner surface to the outer one.

The small thickness of the wall causes the temperature gradient in that direction to be large. Further, if the air temperatures inside and outside the building remain constant, then heat transfer through the wall of a building can be modeled as steady state and one-dimensional. The temperature of the wall in this case depends on one direction only (e.g. the x-direction) and can be expressed as $T(x)$.

The rate of heat transfer given by Fourier law is,

$$\frac{dQ}{dt} = -kA \frac{dT}{dx}$$

$$\text{or, } \frac{dQ}{dt} dx = -kA dT$$

Integrating both sides

$$\int_0^L \frac{dQ}{dt} dx = \int_{T_1}^{T_2} -kA dT$$

$$\text{or, } \frac{dQ}{dT} L = -kA(T_2 - T_1)$$

$$\text{or, } \frac{dQ}{dT} = -\frac{kA(T_2 - T_1)}{L}$$

$$\therefore \frac{dQ}{dT} = \frac{(T_1 - T_2)}{R_{Th}}$$

Where, $R_{Th} = \frac{L}{kA}$ is called thermal resistance.

8.3.2 Convection

Convection is the mode of energy transfer between a solid surface and the adjacent liquid or gas that is in motion, and it involves the combined effects of conduction and fluid motion. The convection heat transfer is possible only in a fluid medium and the heat energy is transferred from the actual movement of the molecules thereby causing molecular collisions and molecular diffusions.

Magnitude of convective heat transfer is given by Newton's law of cooling and it states that

The rate of heat flow per unit area (heat flux) is directly proportional to the temperature difference between a solid surface and an adjacent fluid.

Mathematically, it can be expressed as

$$\frac{dQ}{dt} = hA(T_2 - T_1)$$

where T_2 = temperature of solid surface (K)

T_1 = fluid temperature (K)

h = coefficient of convection heat transfer coefficient

8.3.3 Radiation

Radiation is the transfer of heat energy in the form of electromagnetic wave from one body to another. Unlike the heat transfer by conduction and convection, radiation heat transfer is possible without any medium between the heat source and the receiver. In fact, the heat transfer by radiation or radiation exchange between two bodies occurs most effectively in vacuum where no conduction and convection heat transfer take place. A material medium present between the heat source and the receiver would either reduce or eliminate entirely the propagation of heat energy by radiation.

Stefan-Boltzmann Law of Heat Radiation

Magnitude of the radiation heat transfer by a surface is given by Stefan-Boltzmann law which states that

The maximum rate of heat radiated per unit area (heat flux) by a surface is directly proportional to the fourth power of the absolute surface temperature.

Mathematically,

$$\frac{dQ}{A dt} \propto T^4$$

$$\text{or, } \frac{dQ}{dt} = \sigma A T^4$$

Where σ is a constant of proportionality, known as Stefan-Boltzmann constant its value is $5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$

An ideal surface which emits heat flux at maximum rate as per above equation is called a black body. The heat flux emitted by all real body surface is less than that by a black body surface at same temperature by a factor and expressed as,

$$\frac{dQ}{dt} = \epsilon \sigma A T^4$$

Where ϵ is radiative property of surface and known as emissivity. Its value is one for black body and less than one for all real body.

So, magnitude of rate of radiation heat transfer between two practical bodies at temperature T_1 and T_2 ($T_1 > T_2$) is given as

$$\frac{dQ}{dt} = \epsilon \sigma A (T_1^4 - T_2^4)$$

Emissive power:

Emissive power is defined as the total amount of heat radiation emitted by a body per unit area per unit time.

According to Stefan Boltzmann law, emissive power is,

$$E = \sigma T^4$$

From above equation, $E \propto T^4$

The SI unit of emissive power is Watt/m^2 which gives the intensity of radiation.

Emissivity (ϵ):

It is defined as the ability of surface to radiate heat. Mathematically it can also be defined as the ratio of emissive power of anybody to the emissive power of a black body. For a perfect black body its value is one. The emissivity of polished copper and polished aluminum, are 0.018 and 0.05 further the emissivity of a body is equal to the absorptance of its surface.

SOLUTION TO EXAM PROBLEMS AND IMPORTANT QUESTIONS

1. A closed metal vessel contains water at 30°C . the vessel has surface area 0.5m^2 and thickness 4mm . calculate the heat loss per minute by conduction if the outside temperature is 15°C and thermal conductivity of metal is $400\text{Wm}^{-1}\text{K}^{-1}$ [2022 Fall]

$\Rightarrow$ Here, Surface area (A) = 0.5m^2

$$\text{Temperature difference } (T_2 - T_1) = (30 - 15) \\ = 15^\circ\text{C}$$

$$\text{Thickness of vessel } (L) = 4 \text{ mm} \\ = 0.004 \text{ m}$$

$$\text{Thermal conductivity } (K) = 400 \text{ Wm}^{-1}\text{K}^{-1}$$

$$\text{Rate of heat loss } \left(\frac{dQ}{dt} \right) = ?$$

We know the relation

$$\frac{dQ}{dt} = \frac{kA(T_2 - T_1)}{L}$$

$$= \frac{400 \times 0.5 \times 15}{0.004}$$

$$\therefore \frac{dQ}{dt} = 750 \text{ kJ min}^{-1}$$

2. Calculate the maximum amount of heat which may be lost per second by radiation from a sphere of radius 10cm in diameter at temperature of 227°C when placed in an enclosure at a temperature of 27°C .

$$(\sigma = 5.7 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4})$$

$$\Rightarrow \text{Here, Surface area } (A) = 4\pi r^2 \\ = 4\pi \times (0.05)^2$$

$$\text{Body temperature } (T_2) = 227 + 273 \\ = 500\text{K}$$

$$\text{Surrounding temperature } (T_1) = 27 + 273 \\ = 300\text{K}$$

$$\text{Rate of heat loss } \left(\frac{dQ}{dt} \right) = ?$$

We know the relation Stefan Boltzmann relation

$$\frac{dQ}{dt} = \sigma A(T_2^4 - T_1^4)$$

$$= 5.7 \times 10^{-8} \times 4\pi \times (0.05)^2 \times (500^4 - 300^4) \\ = 97.86\text{J/s.}$$

$$= \frac{400 \times 0.5 \times 15}{0.004}$$

$$\therefore \frac{dQ}{dt} = 750 \text{ kJ min}^{-1}$$

3. One gram molecule of a perfect gas expands isothermally to four times its initial volume at temperature 20°C . Calculate the work done.

$\Rightarrow$ Here, initial volume (V_1) = V

$$\text{Final volume } (V_2) = 4V$$

$$\text{Temperature } (T) = 20 + 273 \\ = 293\text{K}$$

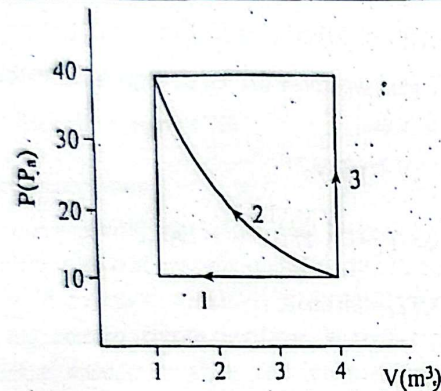
We have, work done in isothermal process,

$$W = RT \ln \left(\frac{V_2}{V_1} \right)$$

$$= 8.4 \times 293 \times \ln 4$$

$$= 3411.9\text{J}$$

4. A sample of gas consisting of 0.11 mol is compressed from a volume of 4 m^3 to 1 m^3 while its pressure increases from 10 to 40 Pa. Compute the work done along the three different paths shown in figure.



⇒ Path 1 consists of two processes, one at constant pressure followed by another at constant volume.

The work done at constant pressure is given by

$$W = -P(V_2 - V_1) \\ = -10(1 - 4) = 30 \text{ J}$$

The work done at constant volume is zero.

So, the total work for path 1 is

$$W_1 = 30 \text{ J}$$

Path 2 consists of isothermal process, so work done at isothermal process is

$$W_2 = -P_1 V_1 \ln \frac{V_2}{V_1} \\ = -10 \times 4 \times \ln \frac{1}{4}$$

$$\therefore W_2 = 55 \text{ J}$$

Path 3 consists of a process at constant volume, for which the work is again zero, followed by a process at constant pressure. So, the total work for path 3 is

$$W_3 = 0 - P_2(V_2 - V_1) \\ = -40(1 - 4)$$

$$\therefore W_3 = 120 \text{ J}$$

5. Find the rate of flow of heat through a square iron plate of each side equal to 4cm and thickness 5mm. its opposite faces are kept at 90°C and 40°C respectively. Thermal conductivity for iron is $63 \text{ W m}^{-1} \text{ K}^{-1}$

$$\Rightarrow \text{Here, Surface area (A)} = 0.04 \times 0.04 \\ = 0.0016 \text{ m}^2$$

$$\text{Temperature difference } (T_2 - T_1) = (90 - 40) \\ = 50^\circ\text{C}$$

$$\text{Thickness of wall (L)} = 5 \text{ mm} \\ = 0.005 \text{ m}$$

$$\text{Thermal conductivity (K)} = 63 \text{ W m}^{-1} \text{ K}^{-1}$$

$$\text{Rate of heat transfer } \left(\frac{dQ}{dt} \right) = ?$$

We know the relation

$$\frac{dQ}{dt} = \frac{kA(T_2 - T_1)}{L} \\ = \frac{63 \times 0.0016 \times 50}{0.005}$$

$$\therefore \frac{dQ}{dt} = 1008 \text{ Watt}$$

Hence the rate of heat transfer is 1008 Watt.

6. A brick wall 12cm thick and 5m^2 surface area is exposed to 250°C at one face and 50°C to another face. If the thermal conductivity of the material is 1.5W/mK , Determine the heat transfer rate. [2023 Spring]

$\Rightarrow$ Here, surface area (A) = 5m^2

$$\text{Temperature difference } (T_2 - T_1) = (250 - 50) \\ = 200^\circ\text{C}$$

$$\text{Thickness of wall } (L) = 12\text{cm} \\ = 0.12\text{m}$$

$$\text{Thermal conductivity } (K) = 1.5\text{Wm}^{-1}\text{K}^{-1}$$

$$\text{Rate of heat transfer } \left(\frac{dQ}{dt}\right) = ?$$

We know the relation

$$\frac{dQ}{dt} = \frac{kA(T_2 - T_1)}{L}$$

$$= \frac{1.5 \times 5 \times 200}{0.12}$$

$$\therefore \frac{dQ}{dt} = 12500 \text{ Watt}$$

Hence the rate of heat transfer is 12500 Watt.

7. An ideal gas at 17°C has pressure of 760 mm of Hg is compressed adiabatically until its volume is halved. Calculate the pressure and temperature of gas. [2023 Spring]

$\Rightarrow$ Here, initial volume (V_1) = V

$$\text{Final volume } (V_2) = \frac{V}{2}$$

$$\text{Initial temperature } (T_1) = 17 + 273 \\ = 290\text{K}$$

$$\text{Final temperature } (T_2) = ?$$

$$\text{Initial pressure } (P_1) = 760\text{mm of Hg}$$

Final Pressure (P_2) = ?

We have, for adiabatic process,

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

Taking $\gamma = 1.4$,

$$\text{or, } P_2 = P_1 \left(\frac{V_1}{V_2}\right)^{1.4} \\ = 760 \times 2^{1.4}$$

$$\therefore P_2 = 2005.65 \text{ mm of Hg.}$$

Again we know the temperature volume relation in adiabatic process,

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

$$\text{or, } T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{1.4-1} = 290 \times 2^{0.4}$$

$$\therefore T_2 = 382.65\text{K}$$

SYLLABUS

Faculty of Science and Technology, Pokhara University
Course Title: Applied Physics (3-1-2) Full marks: 100
Pass Marks: 45

Nature of the Course: Theory and Practical

Time per Period: 1 hour

Year: First Total Periods: 45

Level: Bachelor Program: BE

1. Course Description

This course covers the fundamental topics of physics and basic principles that are required to study other engineering courses. It develops the ability to identify, formulate and solve engineering physics problems. Moreover, it enables to formulate, conduct, analyze and interpret experiments in engineering physics through tutorials, laboratory work and self-learning activities.

2. General Objectives

The general objectives of this course are:

- To equip the students with the fundamental concept and laws of oscillation, electromagnetism and thermodynamics.
- To acquaint the students with waves, laser, optical fiber, quantum mechanics and enlighten the importance of capacitor and dielectrics.

3. Methods of Instruction

Lecture, discussion, tutorials, Laboratory work and assignments

4. Contents in Detail

Specific Objectives	Contents
Understand mechanical oscillations, solve problems related to different types of oscillation, familiar with the scope in various engineering fields.	Unit I: Mechanical Oscillation (6 hrs) 1.1 Free oscillation, Damped oscillation and Forced oscillation (Physical meaning and equations). 1.2 Compound pendulum, Minimum and maximum time period in compound pendulum, Interchangeability of point of suspension and point of oscillation in compound pendulum, Torsion pendulum, Determination of modulus of rigidity of material using torsion pendulum.
Solve and analyze the problems related to waves	Unit II: Wave Motion (4 hrs) Introduction of wave, wave velocity and particle velocity, types of waves and their applications, Speed of wave in stretched string, energy, power and intensity of plane progressive wave, standing wave and resonance, sonometer.
Solve the problems related to reverberation in different units of building. Solve the problems related to ultra sound.	Unit III: Acoustics (4 hrs.) 3.1 Classification of sound waves, Acoustics of building, Reverberation of sound, absorption coefficient, Noise pollution and its control, Sound insulation, Sabine equation. 3.2 Introduction, production and applications of ultrasonic wave. Ultrasonic method in nondestructive testing.

Specific Objectives	Contents
Understand the use of lasers in engineering sciences and solve problems related to laser and fiber optics. Apply the concept of optical fibers in communication system and sensors.	Unit IV: Photonics (6 hrs.) 4.1 Laser: Introduction of laser, Principles of generation of laser light (induced absorption, spontaneous emission, stimulated emission, population inversion, pumping, metastable state), He-Ne laser, Semiconductor laser, Applications of laser. 4.2 Fiber optics: Introduction, Types of optical fiber, Principle of propagation of light wave through optical fiber (Acceptance angle), Numerical aperture, Applications of optical fiber in communications, Optical fiber sensors.
Evaluate the capacity of capacitors to store energy with and without dielectrics. Solve problems related to electrostatics.	Unit V: Capacitor and Dielectric (6 hrs.) 5.1 Capacitor: Introduction, Types of capacitors, Charging and discharging of capacitor. 5.2 Dielectric: Introduction, Dielectric constant, electric flux density, Polarization, Polarization in free space, Gauss law in dielectric, Electronic and Ionic polarization (Clausius-Mossotti equation).
Deal with interaction between electric field and magnetic field on matter.	Unit VI: Electromagnetism (6 hrs.) 6.1 EM Oscillation: LC oscillation, Damped LCR oscillation, Forced electromagnetic oscillation, resonance and quality factor

Specific Objectives	Contents
Analyze the relationship between electric field, magnetic field and speed of wave.	6.2 EM waves: Maxwell equations in integral form, Conversion of Maxwell's equations in differential form, Continuity equation, Relation between electric field, magnetic field and speed of light, wave equations in free space, verification of light wave as an electromagnetic wave, Wave equation in dielectric medium
Apply principles of quantum mechanics to investigate the observables on known wave functions. Solve the problems related to particle wave using Schrodinger's wave equations.	Unit VII: Quantum Mechanics (5 hrs.) Inadequacy of classical mechanics, Importance of quantum mechanics, Matter wave (de-Broglie equation), Wave function and its significance, Energy and momentum operator, Time independent and time dependent Schrodinger wave equations, Application of Schrodinger wave equation for the electron in metal, Normalized wave function describing the motion of an electron inside in an infinite potential well.
Acquainted with the laws of thermodynamics and applications. Solve the problems related to thermodynamics and heat transfer.	Unit VIII: Fundamentals of Thermodynamics and Heat Transfer (8 hrs.) 8.1 Concepts and definition: applications of thermodynamics, properties and state of substance, thermodynamics properties and types, processes (definition, characteristics and examples): reversible and irreversible process.

Specific Objectives	Contents
	8.2 Laws of thermodynamics: First law of thermodynamics, first law for closed system, internal and stored energy, joules law, enthalpy, specific heat, application of first law for closed system, Related problems on closed system, second law of thermodynamics, heat engine (four components of refrigerator and heat pump, COP of refrigerator and heat pumps), Kelvin-Planck and Clausius statement of second law. 8.3 Heat transfer: Modes of heat transfer (conduction, convection and radiation), statement and assumption of Fourier's law of thermal conductivity, one-dimensional steady state heat conduction through plane wall, basic laws of radiation (Emissive power and emissivity, Stefan-Boltzmann's law), Concept of black bodies.

5. Tutorials

1. Solving the problems related to different oscillation.
2. Solving and analyzing the problems related to waves.
3. Determination of standard reverberation time for normal human ear and solving problems related to ultra sound.
4. Determination of angle of acceptance for working of optical fiber and finding population of atoms in different energy states.

5. Solving the problems for different combination of capacitors and finding the charging and discharging time constant for capacitor.
6. Solving the problems related to Gauss law of electrostatics.
7. Determination of frequency of damped and undamped LC oscillation and analyzing the relationship between electric field, magnetic field and speed of wave.
8. Solving the problems related to particle wave using Schrodinger's wave equations.
9. Solving the problems related to thermodynamics and heat transfer.

6. Laboratory Work (Any Eight)

1. To determine the acceleration due to gravity and radius of gyration of bar pendulum.
2. To determine the value of modulus of rigidity of the material given and moment of inertia of circular disc using torsion pendulum.
3. To determine the acceptance angle of an optical fiber using laser source.
4. To determine the frequency of AC mains by using sonometer apparatus.
5. To determine the wavelength of laser light by using diffraction grating.
6. To determine the capacitance of given capacitor by charging and discharging through resistor.
7. To plot a graph between current and frequency in an LRC series circuit and to find:
 - i) the resonance frequency ii) the quality factor.
8. To determine the dielectric constant of a given material.
9. To determine the Planck's constant and photoelectric work functions of the material.

10. To measure the pressure, specific volume and temperature.
11. To find out the efficiency of a compressor.
12. To measure the rate of heat transfer by conduction
13. To measure the performance of a Refrigeration/ Heat pump

7. Evaluation System and Students' Responsibilities

Evaluation System

In addition to the formal exam(s), the internal evaluation of a student may consist of quizzes, assignments, lab reports, projects, class participation, etc. The tabular presentation of the internal evaluation is as follows.

External Evaluation Marks and Internal Evaluation Weight Marks

Semester-End examination 50

Theory 30

Attendance and Class Participation 10%

Assignments 20%

Presentations/Quizzes 10%

Term exam 60%

Practical 20

Attendance and Class Participation 10%

Lab Report/ Project Report 20%

Practical Exam/ Project Work 40%

Viva 30%

Total Internal 50

Full Marks: $50 + 50 = 100$

Student Responsibilities

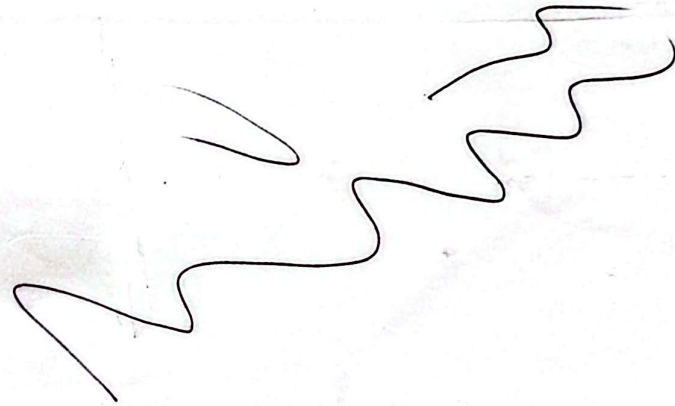
Each student must secure at least 45% marks in internal evaluation with 80% attendance in the class in order to appear in the Semester End Examination. Failing to get such score will be given NOT QUALIFIED (NQ) and the student will not be eligible to appear the Semester-End Examinations. Students are advised to attend all the classes, formal exam, test, etc. and complete all the assignments within the specified time period. Students are required to complete all the requirements defined for the completion of the course.

8. Prescribed Books and References

Text Books

1. Halliday, D., Resnick, R., & Walker, J. *Fundamental of Physics*. John Wiley and Sons. Inc.
2. Young, H. D. & Freedman. R. A. *Sears and Zemansky's University Physics*. 2009.
3. Howel, J. R. & Buckius, R. O. *Fundamentals of Engineering Thermodynamics*. McGraw- Hill Publishers.

Thank you.
Best of luck for exams.



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(I can do)³