

Unit-3

[done] ✓
Laplace Transfer

[8+7+3]

Date _____
Page _____

Define Laplace transfer.

⇒ A function $f(t)$ has Laplace transfer, if it is obtained by multiplication of e^{-st} with $f(t)$ and integrating over the limit $(0, \infty)$ if it exists.

It is denoted by, $L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

i.e. $L[f(t)] = [f(s)]$.

Note: i. $e^{-\infty} = 0$

ii. $e^{\infty} = \infty$

iii. $e^0 = 1$

For example:

Q. Find the L.T. of $t^0 = 1$

⇒ $f(t) = 1$

By the definition,

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$\Rightarrow L[1] = \int_0^{\infty} e^{-st} (1) dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^{\infty} = 0 - \left(\frac{1}{-s} \right) = \frac{1}{s}$$

1. $f(t) = t$

⇒ We have by defⁿ

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$\Rightarrow L[t] = \int_0^{\infty} e^{-st} t dt$$

$$= \left[t \int e^{-st} - \int \left[\frac{dt}{dt} \cdot \int e^{-st} dt \right] dt \right]_0^{\infty}$$

$$= \left[t \frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s} dt \right]_0^{\infty}$$

$$= \left[t \frac{e^{-st}}{-s} - \frac{e^{-st}}{(-s)(-s)} \right]_0^{\infty}$$

$$= \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{\infty}$$

$$= 0 - 0 - \left(0 - \frac{1}{s^2} \right)$$

$$= \frac{1}{s^2} \quad \#$$

Hence, $f(t) = t = \frac{1}{s^2} \quad \#$

formula:

$$\int u \cdot v dx = u \left[\int v dx - \int \left[\frac{du}{dx} \int v dx \right] dx \right]$$

Rule: **ILATE**

Q. find the L.T of t^2 and t^3 by using definition.

⇒ $f(t) = t^2$

We have by defⁿ,

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

⇒ $L[t^2] = \int_0^{\infty} e^{-st} t^2 dt$

$$= \left[t^2 \int e^{-st} dt - \int \left[\frac{dt^2}{dt} \int e^{-st} dt \right] dt \right]_0^{\infty}$$

$$= \left[t^2 \cdot \frac{e^{-st}}{-s} - \int 2t \cdot \frac{e^{-st}}{-s} dt \right]_0^{\infty}$$

$$= \left[t^2 \cdot \frac{e^{-st}}{-s} + \frac{2}{s} \int t \cdot e^{-st} dt \right]_0^{\infty}$$

$$\begin{aligned}
&= \left[t^2 \frac{e^{-st}}{-s} + \frac{2}{s} \left[t \int e^{-st} dt - \int \frac{dt}{dt} \int e^{-st} dt \right] \right]_0^\infty \\
&= \left[t^2 \frac{e^{-st}}{-s} + \frac{2}{s} \left[t \cdot \frac{e^{-st}}{-s} - \int \frac{e^{-st}}{-s} dt \right] \right]_0^\infty \\
&= \left[t^2 \frac{e^{-st}}{-s} + \frac{2}{s} \left[t \cdot \frac{e^{-st}}{-s} + \frac{1}{s} \int e^{-st} dt \right] \right]_0^\infty \\
&= \left[t^2 \frac{e^{-st}}{-s} + \frac{2}{s} \left[t \cdot \frac{e^{-st}}{-s} + \frac{1}{s} \cdot \frac{e^{-st}}{(-s)} \right] \right]_0^\infty \\
&= \left[t^2 \frac{e^{-st}}{-s} - \frac{2t \cdot e^{-st}}{s^2} - \frac{2}{s^3} e^{-st} \right]_0^\infty \\
&= 0 - \left(0 - \frac{2}{s^3} \right) \\
&= \frac{2}{s^3} \#
\end{aligned}$$

Successive integration:

$$\int u \cdot v dx = u \cdot v_1 - u' v_2 + u'' v_3 - u''' v_4 + \dots$$

Where, $v =$ integration (I L A T E)
 $u =$ derivative.

$$\begin{aligned}
8. \quad L[f(t)] &= \int_0^\infty e^{-st} f(t) dt \\
\Rightarrow L(t^3) &= \int_0^\infty e^{-st} t^3 dt.
\end{aligned}$$

$$= \left[t^3 \frac{e^{-st}}{-s} - 3t^2 \frac{e^{-st}}{s^2} + 6t \frac{e^{-st}}{-s^2} - 6 \frac{e^{-st}}{(-s)^3} \right]_0^\infty$$

$$= 0 - \left(0 - \frac{6}{s^3} \right)$$

$$= \frac{6}{s^3} \#$$

Review of derivative and integration.

Derivative.

$$1. \quad \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$2. \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$3. \quad \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

4. $\frac{\partial f}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}$

5. $f_{xy} = f_{yx}$ 6. $\frac{d x^n}{dx} = n x^{n-1}$

7. $\frac{d(u \cdot v)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

8. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

9. chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

Exact diff. eqn:

$M dx + N dy = 0,$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

11. $\frac{d e^x}{dx} = e^x$

12. $\frac{d e^{ax}}{dx} = a e^{ax}$

13. $\frac{d \ln x}{dx} = 1/x$

14. $\frac{d \sin x}{dx} = \cos x$

15. $\frac{d \sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$

16. $\frac{d \cos x}{dx} = -\sin x$

17. $\frac{d \tan x}{dx} = \sec^2 x$

18. $\frac{d \sec x}{dx} = \sec x \cdot \tan x$

19. $\frac{d \cot x}{dx} = -\operatorname{cosec}^2 x$

20. $\frac{d \operatorname{cosec} x}{dx} = -\operatorname{cosec} x \cdot \cot x$

Antiderivative:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$

2. $\int dx = x + C$

3. $\int d(\sin x) = \sin x + C$

4. $\int 1/x dx = \ln(x) + C$

5. $\int e^x dx = e^x + C$

6. $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

7. $\int \sin ax dx = -\frac{\cos ax}{a} + C$

8. $\int \sin x dx = -\cos x + C$

9. $\int \cos x dx = \sin x + C$

10. $\int \tan x dx = \ln(\sec x) + C$

11. $\int \cot x dx = \ln(\sin x) + C$

12. $\int \sec x dx = \ln(\sec x + \tan x) + C$

13. $\int \operatorname{cosec} x dx = \ln(\operatorname{cosec} x - \cot x) + C$

Now to put in integration.

①

Exponential funcⁿ Power मा x या-या फल आयेगा
तो funcⁿ Put $\frac{1}{x} = e$ ।

Ex. $\int x \cdot e^x dx$ — (i)

put. $x^2 = y$
 $2x dx = dy$
 $x \cdot dx = \frac{dy}{2}$

Now, $\int y \cdot \frac{dy}{2} = e^{\frac{y}{2}} + C = \frac{e^x}{2} + C$

(ii) $\int \sin^m x \cos^n x dx$ को Power गिफ़ funcⁿ को $\frac{1}{2}$ कर
 अन ल्यो Power गिफ़को funcⁿ को $\frac{1}{2}$ कर
 Ex. $\int \sin^2 x \cos^6 x dx$ — (i)

put $\cos x = y$

(iii) Trigonometry funcⁿ को angle मा $\frac{1}{2}$ कर $\frac{1}{2}$ कर
 Ex. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$ — (i) the put $\sqrt{x} = y$

i. $a^2 - x^2 \Rightarrow$ put, $x = a \sin \theta$

ii. $a^2 + x^2 \Rightarrow$ put, $x = a \tan \theta$

iii. $x^2 - a^2 \Rightarrow$ put, $x = a \sec \theta$

(a) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$

find L.T of eat

$\Rightarrow$ we have by defⁿ.

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$\Rightarrow L(e^{at}) = \int_0^{\infty} e^{-st} \cdot e^{at} dt$$

$$= \int_0^{\infty} e^{-t(s-a)} dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)} \right]_0^{\infty}$$

$$= 0 - \left(\frac{1}{-(s-a)} \right) = \frac{1}{s-a} \neq$$

Ex $L[e^{3t}] = \frac{1}{s-3} \neq$

* State and prove that Linearity theorem of L.T.

⇒ Statement: Let $f(t)$ and $g(t)$ be two funcⁿ with constant a and b for $t > 0$ then.

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$$

proof:

Here, we have

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt.$$

$$L[af(t) + bg(t)] = \int_0^{\infty} e^{-st} [af(t) + bg(t)] dt.$$

$$= \int_0^{\infty} e^{-st} a f(t) dt + \int_0^{\infty} e^{-st} b g(t) dt.$$

$$= aL[f(t)] + bL[g(t)]$$

proved

Formula: $e^{i\theta} = \cos\theta + i\sin\theta$
 $e^{-i\theta} = \cos\theta - i\sin\theta$

Ex find the L.T of $(at^2 + bt + c)$

$$\Rightarrow f(t) = at^2 + bt + c$$

Taking L.T on both sides,

$$L[f(t)] = L[at^2 + bt + c]$$

$$= aL[t^2] + bL[t] + cL[1]$$

$$= a \cdot \frac{2}{s^3} + b \cdot \frac{1}{s^2} + c \cdot \frac{1}{s}$$

$$= \frac{2a}{s^3} + \frac{b}{s^2} + \frac{c}{s} \quad \#$$

Q. Find the L.T of $\sin at$ and $\cos at$.

⇒ We know that,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

* * *

*. State and prove that Linearity theorem of L.T.

⇒ Statement: Let $f(t)$ and $g(t)$ be two funcⁿ with constant a and b for $t > 0$ then.

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$$

Proof:

Here, we have

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt.$$

$$L[af(t) + bg(t)] = \int_0^{\infty} e^{-st} [af(t) + bg(t)] dt.$$

$$= \int_0^{\infty} e^{-st} a f(t) dt + \int_0^{\infty} e^{-st} b g(t) dt.$$

$$= aL[f(t)] + bL[g(t)]$$

proved

Formula: $e^{i\theta} = \cos\theta + i\sin\theta$
 $e^{-i\theta} = \cos\theta - i\sin\theta$

Ex Find the L.T of $(at^2 + bt + c)$

$$\Rightarrow f(t) = at^2 + bt + c$$

Taking L.T on both sides,

$$L[f(t)] = L[at^2 + bt + c]$$

$$= aL[t^2] + bL[t] + cL[1]$$

$$= a \cdot \frac{2}{s^3} + b \cdot \frac{1}{s^2} + c \cdot \frac{1}{s}$$

$$= \frac{2a}{s^3} + \frac{b}{s^2} + \frac{c}{s} \quad \#$$

Q. Find the L.T of $\sin at$ and $\cos at$.

⇒ We know that,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Rightarrow e^{iat} = \cos at + i \sin at$$

Taking L.T on both sides,

$$L[\cos at + i \sin at] = L[e^{iat}]$$

$$L(\cos at) + i L(\sin at) = \frac{1}{s - ia}$$

$$L(\cos at) + i L(\sin at) = \frac{1}{(s - ia)} \times \frac{s + ia}{(s + ia)}$$

$$L(\cos at) + i L(\sin at) = \frac{s + ia}{s^2 + a^2}$$

$$L(\cos at) + i L(\sin at) = \frac{s}{s^2 + a^2} + \frac{ia}{s^2 + a^2}$$

Equating Real to Real and Imag. to Imag.

$$\Rightarrow L(\cos at) = \frac{s}{s^2 + a^2} \quad \text{Note: } L(\cosh at) = \frac{s}{s^2 - a^2}$$

$$L(\sin at) = \frac{a}{s^2 + a^2} \quad L(\sinh at) = \frac{a}{s^2 - a^2}$$

$$* * * \quad L[f'(t)] = s L[f(t)] - f(0)$$

Laplace transform of derivative.

Let $f(t)$ contin. funcⁿ for $t > 0$, then

$$L[f'(t)] = s L[f(t)] - f(0) \quad \text{for } s > k$$

Proof

We have.

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt.$$

$$\Rightarrow L[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt.$$

$$= \left[e^{-st} f(t) - \int \frac{d}{dt} e^{-st} f(t) dt \right]_0^{\infty}$$

$$= \left[e^{-st} f(t) \right]_0^{\infty} - \int_0^{\infty} -s e^{-st} f(t) dt.$$

$$= 0 - 1 \cdot (f(0)) + s L[f(t)]$$

$$= s L[f(t)] - f(0) \quad \text{proved.}$$

Note: Integration reduced the order of derivative. Ex. $\int y' = y + c$, $\int y'' = y' + c$.

Similarly,

$$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$$

Q. find L.T of $\sin(t+\theta)$, where θ is constant.

$\Rightarrow$ we have.

$$\sin(t+\theta) = \sin t \cos \theta + \cos t \sin \theta$$

Taking L.T on both side

$$L[\sin(t+\theta)] = \cos \theta L(\sin t) + \sin \theta L(\cos t)$$

$$= \cos \theta \cdot \frac{1}{s^2+1} + \sin \theta \cdot \frac{s}{s^2+1}$$

$$= \frac{\cos \theta}{s^2+1} + \frac{s \sin \theta}{s^2+1} \quad \#$$

find the L.T of $\sin at$ by using derivative method.

$\Rightarrow f(t) = \sin at$ — (i)
we have.

$$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0) \quad \text{--- (ii)}$$

$$f(0) = \sin a \cdot 0 = 0$$

$$f'(0) = a \cos at = a$$

$$f''(t) = -a^2 \sin at$$

Now, from eqⁿ (ii) becomes.

$$L[-a^2 \sin at] = s^2 L(\sin at) - s \cdot 0 - a$$

$$-a^2 L(\sin at) = s^2 L(\sin at) - a$$

$$-a^2 L(\sin at) - s^2 L(\sin at) = -a$$

$$- [a^2 L(\sin at) + s^2 L(\sin at)] = -a$$

$$(a^2 + s^2) L(\sin at) = a$$

$$L(\sin at) = \frac{a}{s^2 + a^2} \quad \#$$

Exercise 3A

1. $3t + 4$

$\Rightarrow f(t) = 3t + 4$

Taking L.T. on both side.

$$L[f(t)] = L(3t + 4)$$

$$= 3L(t) + 4L(1)$$

$$= 3 \cdot \frac{1}{s^2} + \frac{4 \cdot 1}{s}$$

$$= \frac{3}{s^2} + \frac{4}{s} \quad \#$$

2. $t^2 + at + b$

$\Rightarrow f(t) = t^2 + at + b$

Taking L.T. on both side.

$$L[f(t)] = L[t^2 + at + b]$$

$$= L(t^2) + aL(t) + bL(1)$$

$$= \frac{2}{s^3} + \frac{a}{s^2} + \frac{b}{s} \quad \#$$

3. $\sin \pi t$

$\Rightarrow f(t) = \sin \pi t$

Taking L.T. on both side.

$$L[f(t)] = L(\sin \pi t)$$

$$= \frac{\pi}{s^2 + \pi^2} \quad \#$$

4. $\sin(\omega t + \theta)$

$\Rightarrow$ We have.

$$\sin(\omega t + \theta) = \sin \omega t \cos \theta + \cos \omega t \sin \theta$$

Taking L.T. on both side

$$L(\sin \omega t + \theta) = L[\sin \omega t \cos \theta + \cos \omega t \sin \theta]$$

$$= \cos \theta L(\sin \omega t) + \sin \theta L(\cos \omega t)$$

$$= \cos \theta \cdot \frac{\omega}{s^2 + \omega^2} + \sin \theta \cdot \frac{s}{s^2 + \omega^2}$$

$$= \frac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2} \quad \#$$

5. $\cos(\omega t + \theta)$

⇒ We have.

$$\cos(\omega t + \theta) = \cos \omega t \cdot \cos \theta - \sin \omega t \cdot \sin \theta$$

Taking L.T on both side.

$$L[\cos(\omega t + \theta)] = L[\cos \omega t \cdot \cos \theta - \sin \omega t \cdot \sin \theta]$$

$$= \cos \theta L(\cos \omega t) - \sin \theta L(\sin \omega t)$$

$$= \cos \theta \cdot \frac{s}{s^2 + \omega^2} - \sin \theta \cdot \frac{\omega}{s^2 + \omega^2}$$

$$= \frac{s \cdot \cos \theta - \omega \sin \theta}{s^2 + \omega^2} \quad \#$$

6. e^{at+b} for $s > a$.

⇒ We have

$$e^{at+b} = e^{at} \cdot e^b$$

Taking L.T on both side

$$L[e^{at+b}] = L[e^{at} \cdot e^b]$$

$$= e^b L[e^{at}]$$

$$= \frac{e^b}{s-a} \quad \#$$

7. $4e^{5t} + 6t^2 - 4\cos 3t + 3\sin 4t$

⇒ $f(t) = 4e^{5t} + 6t^2 - 4\cos 3t + 3\sin 4t$

Taking L.T on both side

$$L(f(t)) = L[4e^{5t} + 6t^2 - 4\cos 3t + 3\sin 4t]$$

$$= 4L(e^{5t}) + 6L(t^2) - 4L(\cos 3t) + 3L(\sin 4t)$$

$$= 4 \cdot \frac{1}{s-5} + 6 \cdot \frac{2}{s^3} - 4 \cdot \frac{s}{s^2+9} + 3 \cdot \frac{4}{s^2+16}$$

$$= \frac{4}{s-5} + \frac{12}{s^3} - \frac{4s}{s^2+9} + \frac{12}{s^2+16} \quad \#$$

8. $\cos^2 t = \frac{1 + \cos 2t}{2}$

⇒ we have

$$\cos^2 t = \frac{1 + \cos 2t}{2}$$

Taking L.T on both side.

$$\begin{aligned} L(\cos^2 t) &= \frac{1}{2} L(1 + \cos 2t) \\ &= \frac{1}{2} [L(1) + L(\cos 2t)] \\ &= \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 4} \right] \end{aligned}$$

9. $\sin^2 t$

⇒ we have,

$$\sin^2 t = \frac{1 - \cos 2t}{2}$$

Taking L.T on both side,

$$\begin{aligned} L(\sin^2 t) &= \frac{1}{2} L(1 - \cos 2t) \\ &= \frac{1}{2} [L(1) - L(\cos 2t)] \\ &= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 4} \right] \end{aligned}$$

$$= \frac{1}{2} \left[\frac{s^2 + 4 - s^2}{s(s^2 + 4)} \right]$$

$$= \frac{2}{s(s^2 + 4)}$$

P.V. state and prove that first shifting theorem of L.T.

⇒ statement: if $L[f(t)] = f(s)$ then $L[e^{at} f(t)] = f(s-a)$ for $s > a$.

proof: let $L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = f(s)$

$$\begin{aligned} \text{then,} \\ L[e^{at} f(t)] &= \int_0^{\infty} e^{-st} e^{at} f(t) dt = \int_0^{\infty} e^{-(s-a)t} f(t) dt \\ &= f(s-a) \end{aligned}$$

proved.

ex. find the L.T of $(e^{2t} \sin 2t)$

⇒ we have,

$$L[e^{at} f(t)] = f(s-a)$$

$$\begin{aligned}
 &\Rightarrow \mathcal{L}[e^{3t} \sin 2t] \\
 &= \mathcal{L}[\sin 2t]_{s \rightarrow (s-3)} \\
 &= \left(\frac{2}{s^2+4} \right)_{s \rightarrow (s-3)} \\
 &= \frac{2}{(s-3)^2+4} \quad \#
 \end{aligned}$$

Q. Find the L.T. of e^{at} by using first shifting theorem.

$$\begin{aligned}
 &\Rightarrow \mathcal{L}(e^{at}) = \mathcal{L}(e^{at} \cdot 1) \\
 &= \mathcal{L}(1)_{s \rightarrow s-a} \\
 &= \left(\frac{1}{s} \right)_{s \rightarrow (s-a)} \\
 &= \frac{1}{(s-a)} \quad \#
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 1. \mathcal{L}[e^{at} \cdot 1] &= \frac{1}{s-a} \\
 2. \mathcal{L}[e^{at} \sin at] &= \left(\frac{a}{s^2+a^2} \right)_{s \rightarrow (s-a)} = \frac{a}{(s-a)^2+a^2} \\
 3. \mathcal{L}[e^{at} \cos at] &= \left(\frac{s}{s^2+a^2} \right)_{s \rightarrow (s-a)} = \frac{s-a}{(s-a)^2+a^2}
 \end{aligned}$$

Q. Find $\mathcal{L.T.} [e^{at} \cos(t+\theta)]$

$\Rightarrow$ We have,

$$\cos(t+\theta) = \cos t \cos \theta - \sin t \sin \theta$$

$$\begin{aligned}
 \mathcal{L}[e^{at} \cos(t+\theta)] &= \mathcal{L}[\cos(t+\theta)]_{s \rightarrow (s-a)} \\
 &= \mathcal{L}[\cos t \cos \theta - \sin t \sin \theta]_{s \rightarrow (s-a)}
 \end{aligned}$$

$$= [\cos \theta \mathcal{L}(\cos t) - \sin \theta \mathcal{L}(\sin t)]_{s \rightarrow (s-a)}$$

$$= \left[\cos \theta \cdot \frac{s}{s^2+1} - \sin \theta \cdot \frac{1}{s^2+1} \right]_{s \rightarrow (s-a)}$$

$$= \frac{(s-a) \cos \theta}{(s-a)^2+1} - \frac{\sin \theta}{(s-a)^2+1}$$

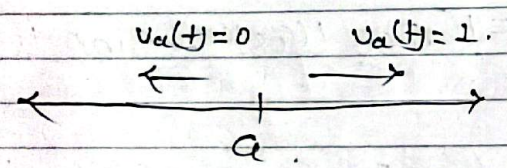
#

* * *

State and prove that Second Shifting theorem.

Define unit step funcⁿ on $\mathbb{C}$.
A funcⁿ $u(t) = u_a(t) = u(t-a)$ is defined by

$$u_a(t) = \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t < a \end{cases} \text{ is called unit step funcⁿ.}$$



* * *

Statement: Let $L[f(t)] = F(s)$ then

$$L[f(t-a) u(t-a)] = e^{-as} F(s) \quad (s > a)$$

proof: Since $L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) \cdot dt$

Then.

$$\begin{aligned} \text{by RHS: } e^{-as} F(s) &= e^{-as} \int_0^{\infty} e^{-s\tau} f(\tau) d\tau \\ &= \int_0^{\infty} e^{-s(a+\tau)} f(\tau) d\tau \end{aligned}$$

put $t = a + \tau$

$$\frac{dt}{d\tau} = 1$$

$$dt = d\tau$$

When $\tau = 0 \Rightarrow t = a$

$\tau = \infty \Rightarrow t = \infty$

Hence,

$$e^{-as} F(s) = \int_a^{\infty} e^{-st} f(t-a) dt$$

$$= 0 + \int_a^{\infty} e^{-st} f(t-a) dt$$

$$= \int_0^a e^{-st} f(t-a) u(t-a) dt + \int_a^{\infty} e^{-st} f(t-a) u(t-a) dt$$

$$= \int_0^{\infty} e^{-st} [f(t-a) u(t-a)] dt$$

$$= L[f(t-a) u(t-a)]$$

proved.

$$e^{-6} e^{-2s} = e^{-6+(-2s)} = e^{-6-2s}$$

$e^{-6} \rightarrow$ first shifting - indicator
 $u_0(t) \rightarrow$ second shifting indicator.

For example:

Q. Find the L.T of $\frac{1}{2}(t) e^{-3t}$

$\Rightarrow$ We have,

$$L[f(t-a) u(t-a)] = e^{-as} f(s) = e^{-as} L[f(t)]$$

$$\text{Now, } L[e^{-3t} u(t-2)]$$

$$= L[e^{-3(t-2)+6} \cdot u(t-2)]$$

$$= L[e^{-3(t-2)} \cdot e^{-6} u(t-2)]$$

$$= e^{-6} \cdot L[e^{-3(t-2)} u(t-2)]$$

$$= e^{-6} \times e^{-2s} \times L[e^{3t}]$$

$$= e^{-6-2s} \cdot \frac{1}{s+3}$$

$$= \frac{e^{-6-2s}}{s+3}$$

Q. $L[\sin 2t u(t)]$

$\Rightarrow$ We have,

$$L[f(t-a) u(t-a)] = e^{-as} L[f(t)]$$

Now,

$$L[\sin 2t u(t-\pi)]$$

$$L[\sin\{2(t-\pi)+2\pi\} u(t-\pi)]$$

$$L[\sin$$

* *

Derivative of Laplace transform. $[q+q]$
Let $f'(t)$ be the derivative of $f(t)$
for $t > 0$, then $L[f'(t)] = s L[f(t)] - f(0)$

show that ① $L[t f(t)] = (-1) \frac{d f(s)}{ds} = -f'(s)$

$$\textcircled{2} L[t^2 f(t)] = (-1)^2 \frac{d^2 f(s)}{ds^2} = (-1) f''(s)$$

$\Rightarrow$ We know that

$$L[f(t)] = f(s) = \int_0^{\infty} e^{-st} f(t) dt$$

Differentiate $f(s)$ w.r. to s .

$$\frac{d f(s)}{ds} = \int_0^{\infty} \frac{d e^{-st}}{d(-st)} \times \frac{d(-st)}{ds} \cdot f(t) dt$$

$$\Rightarrow f'(s) = \int_0^{\infty} e^{-st} (-t) f(t) dt$$

$$\Rightarrow (-1) f'(s) = \int_0^{\infty} e^{-st} [t f(t)] dt$$

$$\Rightarrow (-1) f'(s) = L[t f(t)]$$

proved.

Ex. find the L.T. of $L[-t e^{2t}]$

$\Rightarrow$ We know that,

$$L[-t f(t)] = -f'(s)$$

$$\Rightarrow L[-t e^{2t}] = - \frac{d L[e^{2t}]}{ds}$$

$$= - \frac{d}{ds} \left[\frac{1}{s-2} \right]$$

$$= - \frac{d (s-2)^{-1}}{ds}$$

$$= - \frac{d (s-2)^{-1}}{d(s-2)} \times \frac{d(s-2)}{ds}$$

$$= (s-2)^{-2} \cdot 1$$

$$= \frac{1}{(s-2)^2} \quad \text{H}$$

Q. $L[t \sin 4t]$

⇒ we have,

$$L[t f(t)] = - \frac{d f(s)}{ds} = - \frac{d [L f(t)]}{ds}$$

$$\Rightarrow L[t \sin 4t] = - \frac{d [L(\sin 4t)]}{ds}$$

$$= - \frac{d \left[\frac{4}{s^2 + 16} \right]}{ds}$$

$$= -4 \frac{d \left[\frac{1}{s^2 + 16} \right]}{ds}$$

$$= -4 \frac{d [s^2 + 16]^{-1}}{ds}$$

$$= -4 \frac{d(s^2 + 16)^{-1}}{d(s^2 + 16)} \times \frac{d(s^2 + 16)}{ds}$$

$$= 4 (s^2 + 16)^{-2} \cdot 2s$$

$$= \frac{8s}{(s^2 + 16)^2} \quad \#$$

Note: If a func is +ve or multiply sign
then derivative but divide sign in Antideriva
Integration of the func $f(t)$ after the
derivative of t .

$$L \left[\frac{f(t)}{t} \right] = \int_s^\infty f(\bar{s}) d\bar{s}$$

Example:

Find L.T of $L \left[\frac{\sin 2t}{t} \right]$

⇒ we have. $L \left[\frac{f(t)}{t} \right] = \int_s^\infty f(\bar{s}) d\bar{s}$

$$\Rightarrow L \left[\frac{\sin 2t}{t} \right] = \int_s^\infty L(\sin 4) d\bar{s}$$

$$= \int_s^\infty \frac{2}{\bar{s}^2 + 4} d\bar{s}$$

$$= 2 \int_s^\infty \frac{d\bar{s}}{\bar{s}^2 + 2^2}$$

$$= 2 \left[\frac{1}{2} \tan^{-1} \left(\frac{\bar{s}}{2} \right) \right]_s^\infty$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1} \left(\frac{s}{2} \right) \right]$$

$$= \frac{\pi}{2} - \tan^{-1} \left(\frac{s}{2} \right) \quad \#$$

Assignment-1:

Unit-3 Laplace transform

Exercise - 3A - F



A. Find the Laplace transform of the following $f(t)$ where a, b, T, ω, θ are constant.

1. $3t + 4$

$\Rightarrow f(t) = 3t + 4$

Taking L.T. on both side.

$$L f(t) = L(3t + 4)$$

$$= 3L(t) + 4L(1)$$

$$= 3 \cdot \frac{1}{s^2} + 4 \cdot \frac{1}{s}$$

$$= \frac{3}{s^2} + \frac{4}{s} \quad \#$$

2. $t^2 + at + b$

$\Rightarrow f(t) = t^2 + at + b$

Taking L.T. on both side.

$$L f(t) = L[t^2 + at + b]$$

$$= L(t^2) + aL(t) + bL(1)$$

$$= \frac{2}{s^2+1} + a \cdot \frac{1}{s^2+1} + b \cdot \frac{1}{s}$$

$$= \frac{2}{s^3} + \frac{a}{s^2} + \frac{b}{s} \quad \#$$

3. $\sin \pi t$

$\Rightarrow f(t) = \sin \pi t$

Taking L.T. on both sides.

$$= L f(t) = L(\sin \pi t)$$

$$= \frac{\pi}{s^2 + \pi^2} \quad \#$$

4. $\sin(\omega t + \theta)$

$$\Rightarrow \sin(\omega t + \theta) = \sin \omega t \cdot \cos \theta + \cos \omega t \cdot \sin \theta$$

$$f(t) = \sin \omega t \cdot \cos \theta + \cos \omega t \cdot \sin \theta$$

Taking Laplace transform on both sides.

$$L f(t) = L[\sin \omega t \cdot \cos \theta + \cos \omega t \cdot \sin \theta]$$

$$= \cos \theta \mathcal{L}[\sin \omega t] + \sin \theta \mathcal{L}[\cos \omega t]$$

$$= \cos \theta \cdot \frac{\omega}{s^2 + \omega^2} + \sin \theta \cdot \frac{s}{s^2 + \omega^2}$$

$$= \frac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2}$$

5. $\cos(\omega t + \theta)$

$$\Rightarrow f(t) = \cos(\omega t + \theta) = \cos \omega t \cdot \cos \theta - \sin \omega t \cdot \sin \theta$$

Taking $\mathcal{L} \cdot \uparrow$ on both side.

$$\mathcal{L}f(t) = \mathcal{L}[\cos \omega t \cdot \cos \theta - \sin \omega t \cdot \sin \theta]$$

$$= \cos \theta \mathcal{L}[\cos \omega t] - \sin \theta \mathcal{L}[\sin \omega t]$$

$$= \cos \theta \cdot \frac{s}{s^2 + \omega^2} - \sin \theta \cdot \frac{\omega}{s^2 + \omega^2}$$

$$= \frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$$

#

6. e^{at+b} for $s > a$

$$\Rightarrow f(t) = e^{at+b}$$

$$\mathcal{L}f(t) = \mathcal{L}[e^{at+b}]$$

$$= \mathcal{L}[e^{at} \cdot e^b]$$

$$= e^b \mathcal{L}[e^{at}]$$

$$= \frac{e^b}{s-a} \quad \#$$

7. $4e^{5t} + 6t^2 - 4\cos 3t + 3\sin 4t$

$$\Rightarrow f(t) = 4e^{5t} + 6t^2 - 4\cos 3t + 3\sin 4t$$

$$\mathcal{L}f(t) = \mathcal{L}[4e^{5t} + 6t^2 - 4\cos 3t + 3\sin 4t]$$

$$= \mathcal{L}[4e^{5t}] + 6\mathcal{L}[t^2] - 4\mathcal{L}[\cos 3t] + 3\mathcal{L}[\sin 4t]$$

$$= 4 \cdot \frac{1}{s-5} + 6 \cdot \frac{2}{s^3} - 4 \cdot \frac{s}{s^2+9} + 3 \cdot \frac{4}{s^2+16}$$

$$= \frac{4}{s-5} + \frac{12}{s^3} - \frac{4s}{s^2+9} + \frac{12}{s^2+16} \quad \#$$

8. $\cos^2 t$

$$\Rightarrow f(t) = \cos^2 t = \frac{1 + \cos 2t}{2}$$

$$L(f(t)) = L\left[\frac{1 + \cos 2t}{2}\right]$$

$$= \frac{1}{2} [L(1) + L(\cos 2t)]$$

$$= \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 4} \right] \quad \#$$

9. $\sin^2 t$

$$\Rightarrow f(t) = \sin^2 t = \frac{1 - \cos 2t}{2}$$

$$L(f(t)) = L\left[\frac{1 - \cos 2t}{2}\right]$$

$$= \frac{1}{2} [L(1) - L(\cos 2t)]$$

$$= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 4} \right]$$

$$= \frac{1}{2} \left[\frac{s^2 + 4 - s^2}{s(s^2 + 4)} \right] = \frac{2}{s(s^2 + 4)} \quad \#$$

B. Find the Laplace transform of the following

1. $e^{-2t} \cos t$

$\Rightarrow$ We have.

$$L[e^{at} f(t)] = f(s+a)$$

$$= L[e^{-2t} \cos t]$$

$$= L(\cos t)_{s \rightarrow (s+2)}$$

$$= \left[\frac{s}{s^2 + 1} \right]_{s \rightarrow (s+2)}$$

$$= \frac{s+2}{(s+2)^2 + 1} \quad \#$$

2. $e^{-2t} \sin 4t$

$$\Rightarrow L(f(t)) = L[\sin 4t]$$

$$= \frac{4}{s^2 + 16}$$

By first shifting theorem.

We have.

$$L[e^{-at} f(t)] = \phi(s+a)$$

$$L[e^{-2t} \sin 4t] = \left[\frac{4}{s^2 + 16} \right]_{s \rightarrow s+2}$$

$$= \frac{4}{(s+2)^2 + 16}$$

$$= \frac{4}{s^2 + 4s + 4 + 16}$$

$$= \frac{4}{s^2 + 4s + 20}$$

3. $e^{-2t} \sin n\pi t$

$\Rightarrow$ We have.

$$L[e^{-at} f(t)] = \phi(s+a)$$

$$L[e^{-2t} \sin n\pi t] = \left[L(\sin n\pi t) \right]_{s \rightarrow s+2}$$

$$= \left[\frac{n\pi}{s^2 + (n\pi)^2} \right]_{s \rightarrow s+2}$$

$$= \frac{n\pi}{(s+2)^2 + n^2\pi^2}$$

4. $e^{at} \cos \omega t$.

$\Rightarrow$ By first shifting theorem.

$$L[e^{at} f(t)] = \phi(s-a)$$

$$L[e^{at} \cos \omega t] = \left[L(\cos \omega t) \right]_{s \rightarrow s-a}$$

$$= \left[\frac{s}{s^2 + \omega^2} \right]_{s \rightarrow s-a}$$

$$= \frac{s-a}{(s-a)^2 + \omega^2}$$

5. $e^{at} \sin \omega t$

⇒ By first shifting theorem,

$$L[e^{at} f(t)] = \phi(s-a)$$

Now,

$$L[e^{at} \sin \omega t] = [L(\sin \omega t)]_{s \rightarrow s-a}$$

$$= \left[\frac{\omega}{s^2 + \omega^2} \right]_{s \rightarrow s-a}$$

$$= \frac{\omega}{(s-a)^2 + \omega^2} \quad \#$$

6. $e^{-at} (A \cos \theta t + B \sin \theta t)$

⇒ By first shifting theorem,

$$L[e^{-at} f(t)] = \phi(s+a)$$

Now,

$$L[e^{-at} (A \cos \theta t + B \sin \theta t)]$$

$$= [L(A \cos \theta t + B \sin \theta t)]_{s \rightarrow s+a}$$

$$= [A L(\cos \theta t) + B L(\sin \theta t)]_{s \rightarrow s+a}$$

$$= \left[\frac{A \cdot s}{s^2 + \theta^2} + \frac{B \cdot \theta}{s^2 + \theta^2} \right]_{s \rightarrow s+a}$$

$$= \left[\frac{A \cdot s + B \cdot \theta}{s^2 + \theta^2} \right]_{s \rightarrow s+a}$$

$$= \frac{A(s+a) + B \theta}{(s+a)^2 + \theta^2} \quad \#$$

7. $e^{-3t} (2 \cos 5t - 3 \sin 5t)$

⇒ By first shifting theorem,

$$L[e^{-at} f(t)] = \phi(s+a)$$

Now,

$$L[e^{-3t} (2\cos 5t - 3\sin 5t)]$$

$$= \left[L(2\cos 5t - 3\sin 5t) \right]_{s \rightarrow s+3}$$

$$= \left[2L(\cos 5t) - 3L(\sin 5t) \right]_{s \rightarrow s+3}$$

$$= \left[\frac{2 \cdot s}{s^2 + 25} - \frac{3 \cdot 5}{s^2 + 25} \right]_{s \rightarrow s+3}$$

$$= \left[\frac{2s - 15}{s^2 + 25} \right]_{s \rightarrow s+3}$$

$$= \frac{2(s+3) - 15}{(s+3)^2 + 25}$$

$$= \frac{2s + 6 - 15}{s^2 + 6s + 9 + 25}$$

$$= \frac{2s - 9}{s^2 + 6s + 34} \quad \#$$

$$Q. 5e^{2t} \sinh 2t$$

By shifting theorem,

$$L[e^{at} f(t)] = \phi(s-a)$$

Now,

$$L[e^{2t} \cdot 5 \sinh 2t] = \left[L(5 \sinh 2t) \right]_{s \rightarrow s-2}$$

$$= \left[5 L(\sinh 2t) \right]_{s \rightarrow s-2}$$

$$= \left[\frac{5 \cdot 2}{s^2 - 4} \right]_{s \rightarrow s-2}$$

$$= \frac{10}{(s-2)^2 - 4}$$

$$= \frac{10}{s^2 - 4s + 4 - 4}$$

$$= \frac{10}{s^2 - 4s} \quad \#$$

9. $\sin at \sin bt$

$$\Rightarrow f(t) = \sin at \cdot \sin bt$$

$$L f(t) = L [\sin at \cdot \sin bt]$$

$$= L \left[\frac{1}{2} (\cos(at+bt) - \cos(at-bt)) \right]$$

$$= \frac{1}{2} [L(\cos at \cdot \cos bt) - \sin at \sin bt]$$

changing the order of the integration.

1. $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dy dx$ evaluate the integral by changing the order of integration if necessary.

⇒ Given that,

$$x=0 \text{ to } x=\pi$$

$$y=x \text{ to } y=\pi$$

Now, changing the order of integration.

$$\int_0^{\pi} \int_0^y \frac{\sin y}{y} dx dy$$

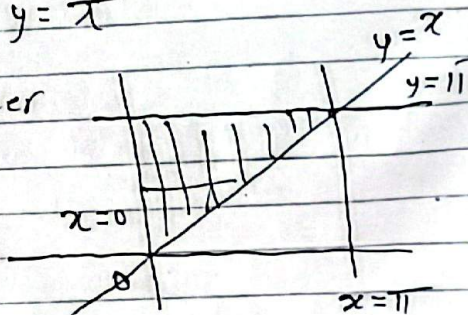
$$= \int_0^{\pi} \frac{\sin y}{y} [x]_0^y dy$$

$$= \int_0^{\pi} \frac{\sin y}{y} \cdot [y-0] dy$$

$$= [-\cos y]_0^{\pi}$$

$$= [-\cos \pi + \cos 0]$$

$$= 1 + 1 = 2 \text{ sq. unit.}$$



Q. evaluate the integral by changing the order of integration.

$$\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} x dx dy.$$

⇒ Solution: Given.

$$\text{Here } y=0 \text{ to } y=\frac{a}{\sqrt{2}}$$

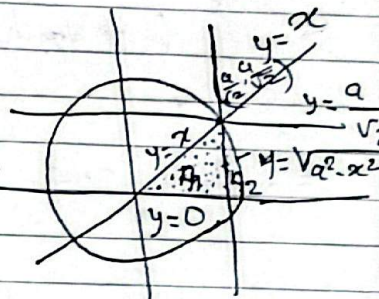
$$x=y \text{ to } x=\sqrt{a^2-y^2} \Rightarrow$$

$$A = A_1 + A_2$$

$$= \int_0^{\frac{a}{\sqrt{2}}} \int_0^x x dy dx + \int_{\frac{a}{\sqrt{2}}}^a \int_0^{\sqrt{a^2-x^2}} x dy dx.$$

$$= \int_0^{\frac{a}{\sqrt{2}}} x [y]_0^x dx + \int_{\frac{a}{\sqrt{2}}}^a x [y]_0^{\sqrt{a^2-x^2}} dx$$

=



How to change Cartesian Coordinate into polar Coordinate?

$\Rightarrow$ Since, $x = r \cos \theta$, $y = r \sin \theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$

$\therefore dx dy = dy dr = r dr d\theta$

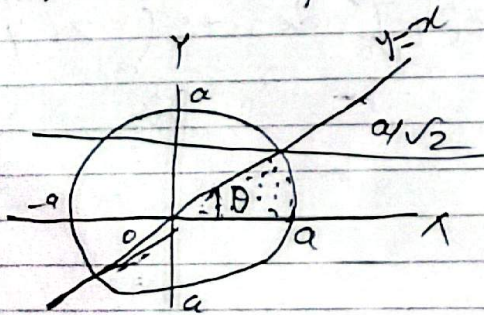
Note: x-axis gives radius
y-axis gives angle

Q. $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2 - y^2}} x dx dy$ by using polar coordinate.

$\Rightarrow$ Given that,

$x = y$ to $x = \sqrt{a^2 - y^2}$

$y = 0$ to $y = a/\sqrt{2}$



By Polar condition,

$x = r \cos \theta$, $y = r \sin \theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$
we have,

$0 \leq \theta \leq \frac{\pi}{4}$
 $0 \leq r \leq a$

then, $A = \int_0^{\pi/4} \int_0^a r \cos \theta r dr d\theta$

$= \int_0^{\pi/4} \int_0^a r^2 \cos \theta dr d\theta$

$= \int_0^{\pi/4} \cos \theta \left[\frac{r^3}{3} \right]_0^a d\theta$

$= \frac{a^3}{3} \int_0^{\pi/4} \cos \theta d\theta$

$= \frac{a^3}{3} [\sin \theta]_0^{\pi/4}$

$= \frac{a^3}{3} \times \frac{1}{\sqrt{2}} = \frac{a^3}{3\sqrt{2}} \text{ sq. unit}$

h.w = Exercise. Q. No. 3 All

Q.1. $\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} y \, dx \, dy$

Q. $\int_0^2 \int_0^{\sqrt{4-y^2}} y \, dx \, dy$

Exercise:

3. change the Cartesian integral into an equivalent polar integral and evaluate the polar integral.

a. $\int_0^2 \int_0^{4-y^2} y \, dx \, dy$

⇒ Given that,

$y=0$ to $y=2$
 $x=0$ to $x=4-y^2$

Ex. 1(A):

Integration $f(x, y)$ over the region R .
 4(a) If $f(x, y) = x$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, $x = 2$.

⇒ Since the curve $y = x$, $y = 2x$, $x = 1$, $x = 2$

Now

$$A = ABCD = \int_1^2 \int_{2x}^x x \, dy \, dx$$

$$= \int_1^2 x \left(\int_{2x}^x \frac{1}{y} \, dy \right) dx$$

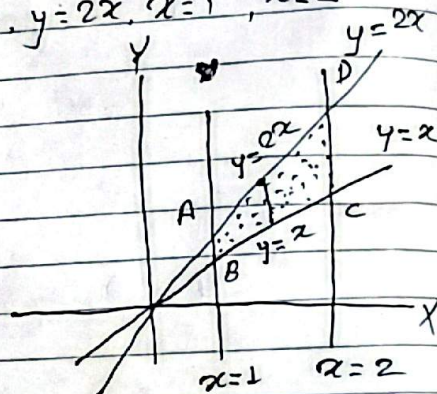
$$= \int_1^2 x \left[\log y \right]_{2x}^x dx \quad \left[\because \frac{1}{y} dy = \log y \right]$$

$$= \int_1^2 x \left[\ln 2x - \ln x \right] dx \quad \left[\because \ln x - \ln y = \ln \left(\frac{x}{y} \right) \right]$$

$$= \int_1^2 x \ln 2 \, dx$$

$$= \ln 2 \left[\frac{x^2}{2} \right]_1^2$$

$$= \ln 2 \left[\frac{4}{2} - \frac{1}{2} \right] = \frac{3}{2} \ln 2 \quad \#$$



Application of double integration. [8 marks]

① Note: Area(A) = $\iint_A f(x, y) \, dy \, dx = \iint_A f(x, y) \, dx \, dy$

② Volume(V) = $\iint z \, dA = \iint z \, dx \, dy = \iint z \, dy \, dx$

eq. find the volume of the object whose base in xy plane, i.e. bounded by the parabola $y = 4 - x^2$ and line $y = 3x$ while the top of the solid object bounded by the plane $z = x + 4$.

⇒ [Given] The solid object is constructed through two curve $y = 4 - x^2$ and $y = 3x$ as shown in the figure.

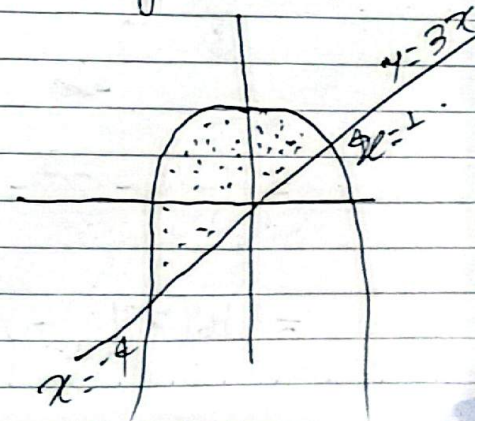
for the limiting value, solving eqn $y = 3x$
 $y = 4 - x^2$

$$\Rightarrow 3x = 4 - x^2$$

$$\Rightarrow x^2 + 3x - 4 = 0$$

$$\Rightarrow x^2 + 4x - x - 4 = 0$$

$$\therefore x = -4, 1$$



required Volume (V) = $\iint z \, dA$

$$= \int_{-4}^4 \int_{3x}^{4-x^2} (4-x^2) \, dy \, dx$$

$$= \int_{-4}^4 (4-x^2) \left[y \right]_{3x}^{4-x^2} dx$$

$$= \int_{-4}^4 (4-x^2) [4-x^2-3x] dx$$

$$= \int_{-4}^4 (16-4x^2-12x-4x^3+3x^2) dx$$

$$= \left[16x - \frac{4x^3}{3} - 6x^2 + \frac{3x^4}{4} \right]_{-4}^4$$

$$= \int_{-4}^4 (16-x^2-16x+x^3) dx$$

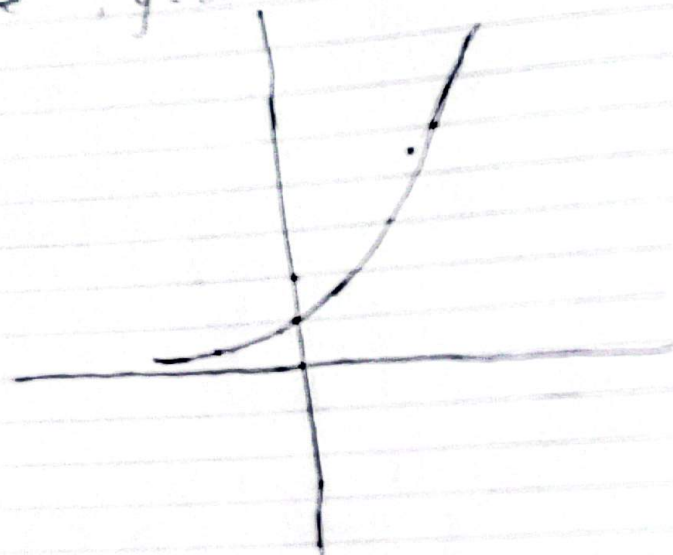
$$= 16 \left[x \right]_{-4}^4 - \left[\frac{x^3}{3} \right]_{-4}^4 - 16 \left[\frac{x^2}{2} \right]_{-4}^4 + \left[\frac{x^4}{4} \right]_{-4}^4$$

$$= 16(1+4) - \frac{1}{3} [1 -$$

114.58

8 Sketch the region bounded by the graph

$$y = e^x, y = \sin x, x = -\pi \text{ to } \pi$$



$$y = e^x$$

x	0	1	2	3	4	5	6	7	8	9	10
y	1	1.7	2.7	4.0	5.5	7.4	10.0	13.5	18.4	24.8	33.9

$$y = \sin x$$

$$A = \int_{-\pi}^{\pi} \int_{\sin x}^{e^x} dy dx$$

$$= \int_{-\pi}^{\pi} [y]_{\sin x}^{e^x} dx$$

$$= \int_{-\pi}^{\pi} (e^x - \sin x) dx$$

$$= [e^x - (-\cos x)]_{-\pi}^{\pi}$$

$$= e^{\pi} \cdot e^{-\pi} - [-\cos \pi + \cos(-\pi)]$$

$$= e^{\pi} - e^{-\pi} - 0$$

$$= e^{\pi} - e^{-\pi}$$

S.M.F

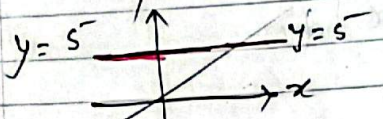
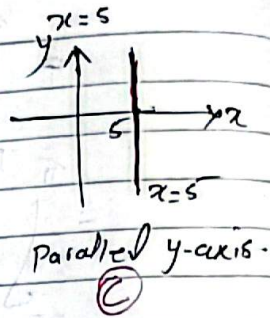
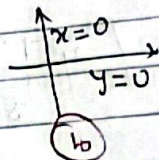
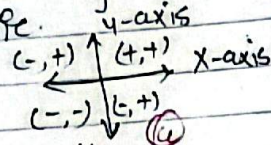
desmos app

Multiple Integration:

Exercise:

To change the order of integral:

for base:



Parallel to x-axis.

Parallel y-axis.

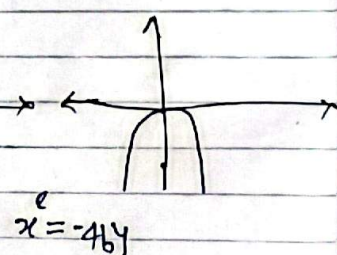
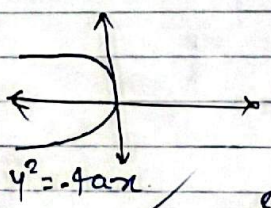
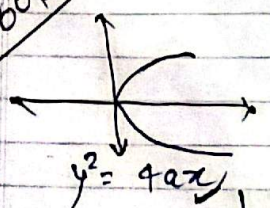
$$y = mx$$

$$y = mx + c = 2x + 2$$

$$y = mx$$

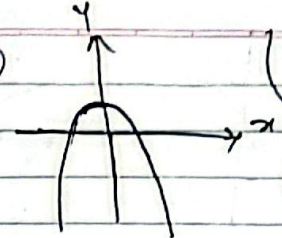
$$x = 4y$$

Parabola

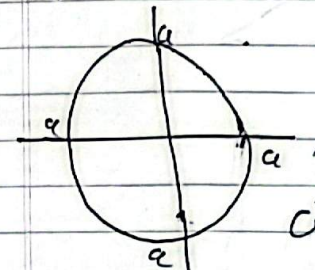
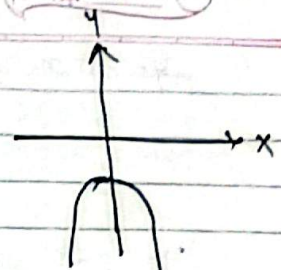


$$x^2 = 4by$$

eg. $x^2 = -y - 2$



$$x^2 = -y - 2$$



Circle

$$\begin{aligned} x^2 + y^2 &= a^2 \\ y^2 &= a^2 - x^2 \\ y &= \sqrt{a^2 - x^2} \end{aligned}$$

$$\begin{aligned} x^2 &= a^2 - y^2 \\ x &= \sqrt{a^2 - y^2} \end{aligned}$$

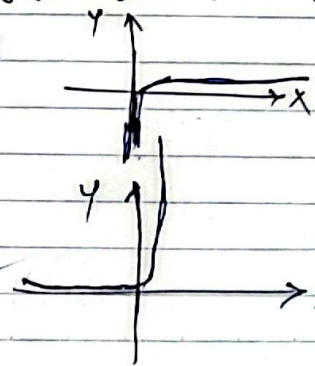
for y को लागि

for dx or x को लागि

logarithmic function

$$y = \log x$$

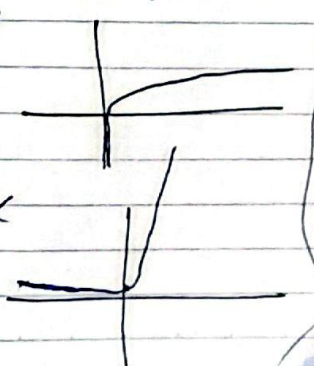
$$x = \log y$$



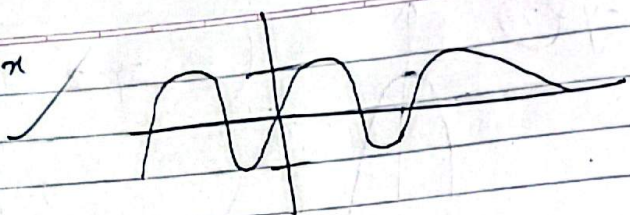
exponential function

$$x = e^y$$

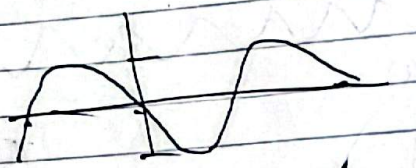
$$y = e^x$$



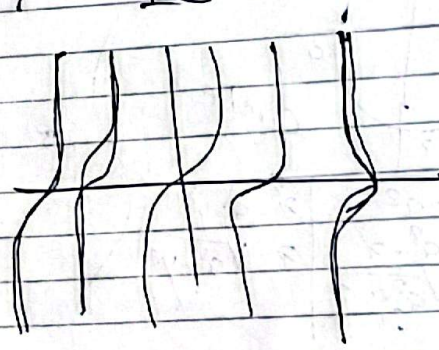
$$y = \sin x$$



$$y = \cos x$$



$$y = \tan x$$



multiple integration:

Exercise: 1[A]

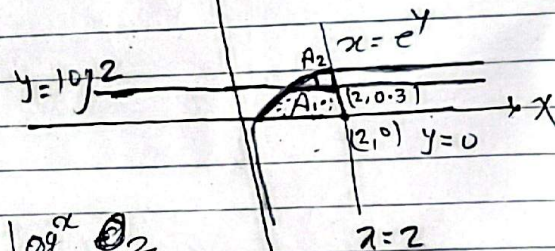
1. Sketch the region and evaluate the following integrates.

$$a \int_0^{\log 2} \int_{e^y}^2 dx dy$$

⇒ Here.

$$x = 2 \text{ to } x = e^y$$

$$y = \log 2 \text{ to } y = 0$$



Now,

$$A = A_1 + A_2$$

$$= \int_0^{\log 2} \int_{e^y}^2 dx dy + \int_0^{\log 2} \int_{e^y}^2 dx dy$$

$$= \int_0^{\log 2} [x]_{e^y}^2 dy + \int_0^{\log 2} [x]_{e^y}^2 dy$$

$$= \int_0^2 \log 2 dy + \int_0^2 e^y dy$$

$$= \log 2 [y]_0^2 + [e^y]_0^2$$

$$= \log 2 [2-0] + [e^2 - e^0]$$

$$= 2 \log(2)$$

$$= \int_0^{\log 2} 2 dy + \int_0^{\log 2} (2 - e^y) dy$$

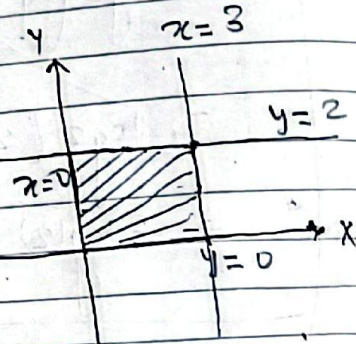
$$= 2 [y]_0^{\log 2} + [2[y] - (e^y)]_0^{\log 2}$$

$$= 2 \log(2) + 2 \cdot \log 2 - e^{\log 2} - 0 - e^0$$

$$= 2 \log(2) - 1$$

b. $\int_0^3 \int_0^2 (4 - y^2) dy dx$

$\Rightarrow$ Here $y=0$ to $y=2$
 $x=0$ to $x=3$



Now,

$$I = \int_0^3 \int_0^2 (4 - y^2) dy dx$$

$$= \int_0^3 \left\{ 4 \left[y \right]_0^2 - \left[\frac{y^3}{3} \right]_0^2 \right\} dx$$

$$= \int_0^3 \left\{ 4 \left(2 - 0 \right) - \left(\frac{2^3}{3} - \frac{0^3}{3} \right) \right\} dx$$

$$= \int_0^3 \left(8 - \frac{8}{3} \right) dx$$

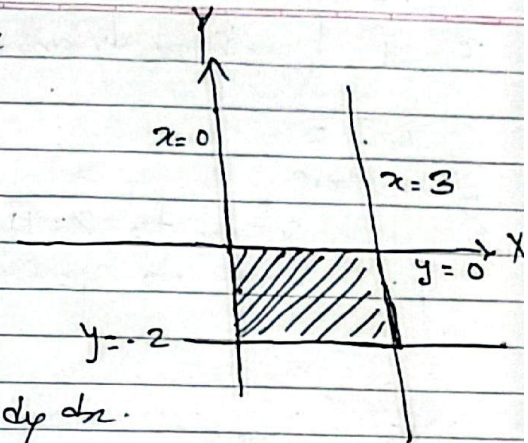
$$= \frac{16}{3} \left[x \right]_0^3$$

$$= \frac{16}{3} [3 - 0] = 16 \text{ sq. unit.}$$

c. $\int_0^3 \int_{-2}^0 (x^2 y - 2xy) dy dx$

$\Rightarrow$ Here,

$y=0$ to $y=-2$
 $x=0$ to $x=3$



Now,

$$I = \int_0^3 \int_{-2}^0 (x^2 y - 2xy) dy dx$$

$$= \int_0^3 \left[\int_{-2}^0 x^2 y dy - \int_{-2}^0 2xy dy \right] dx$$

$$= \int_0^3 \left[x^2 \left[\frac{y^2}{2} \right]_{-2}^0 - 2x \left[\frac{y^2}{2} \right]_{-2}^0 \right] dx$$

$$= \int_0^3 \left[x^2 \left(\frac{0^2}{2} - \frac{4}{2} \right) - 2x \left(\frac{0^2}{2} - \frac{4}{2} \right) \right] dx$$

$$= \int_0^3 (-2x^2 + 4x) dx$$

$$= \left[-2 \left[\frac{x^3}{3} \right] + 4 \left[\frac{x^2}{2} \right] \right]_0^3 = -2 \left(\frac{3^3}{3} - 0 \right) + 4 \left(\frac{3^2}{2} - 0 \right)$$

$$= -18 + 18$$

$$= 0$$

$$I = \int_0^{\pi} \int_0^x x \sin y \, dy \, dx$$

Here
 $x=0$ to $x=\pi$
 $y=0$ to $y=x$



Now

$$\begin{aligned} I &= \int_0^{\pi} \int_0^x x \sin y \, dy \, dx \\ &= \int_0^{\pi} x \left[-\cos y \right]_0^x dx \\ &= \int_0^{\pi} x (-\cos x + \cos 0) dx \\ &= \int_0^{\pi} (-x \cos x + x) dx \end{aligned}$$

$$= \left[-x \sin x - (-\cos x) + \frac{x^2}{2} \right]_0^{\pi}$$

$$\begin{aligned} &= \left[-x \sin x - \cos x + \frac{x^2}{2} \right]_0^{\pi} \\ &= 2 + \frac{\pi^2}{2} = \frac{4 + \pi^2}{2} \end{aligned}$$

$$I = \int_0^{\pi} \int_0^x x \sin y \, dy \, dx$$

Here

$x=0$ to $x=\pi$
 $y=0$ to $y=x$



Now

By the condition,
 $x = \cos 0, y = \sin 0$
 $0 = \cos^{-1}(y)$

We have

$$0 \leq \theta \leq \pi/2$$

$$0 \leq r \leq 2$$

$$I = \int_0^{\pi/2} \int_0^2 r \, dr \, d\theta$$

$$\begin{aligned}
 &= \int_0^a \left(\frac{\pi}{2} \right) dx \\
 &= \left(\frac{\pi}{2} x \right) \Big|_0^a \\
 &= \frac{\pi}{2} a \\
 &= \frac{\pi}{2} (a - 0) = \frac{\pi a^2}{2} =
 \end{aligned}$$

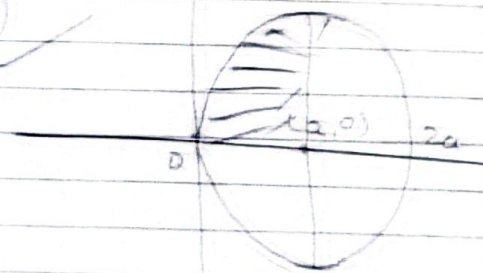
4. Show that the Volume enclosed between the cylinders $(x^2 + y^2 = 2ax)$ and $z = \sqrt{2a^2 - x^2}$ is $\frac{4\pi a^3}{3}$.

$$\begin{aligned}
 \Rightarrow x^2 + y^2 &= 2ax \\
 \Rightarrow x^2 - 2ax + y^2 &= 0
 \end{aligned}$$

$$\begin{aligned}
 x^2 - 2ax + a^2 - a^2 + y^2 &= 0 \\
 (x-a)^2 + (y-0)^2 &= a^2
 \end{aligned}$$

circle with center $(a, 0)$
radius $\frac{1}{2}(2a) = a$

$$\begin{aligned}
 \text{Again, } z^2 &= 2a^2 - x^2 \\
 z &= \sqrt{2a^2 - x^2}
 \end{aligned}$$



$$\text{Required Volume (V)} = \int \int \int dz dy dx = dV$$

$$= 4 \int_0^a \int_0^{\sqrt{2a^2 - x^2}} \sqrt{2a^2 - x^2} dy dx$$

$$= 4 \sqrt{2a^2 - x^2} \int_0^{\sqrt{2a^2 - x^2}} dy dx$$

$$= 4 \sqrt{2a^2 - x^2} \left[y \right]_0^{\sqrt{2a^2 - x^2}} dx$$

$$= 4\sqrt{2a} \int_0^a x \sqrt{2a-x} dx$$

Put $y = 2a - x$
 $dy = -dx$

When $x=0$, $y=2a$
 $x=a$, $y=a$

Now.

$$V = 4\sqrt{2a} \int_{2a}^a (2a-y) \sqrt{y} (-dy)$$

$$= 4\sqrt{2a} \int_a^{2a} (2a-y) \sqrt{y} dy$$

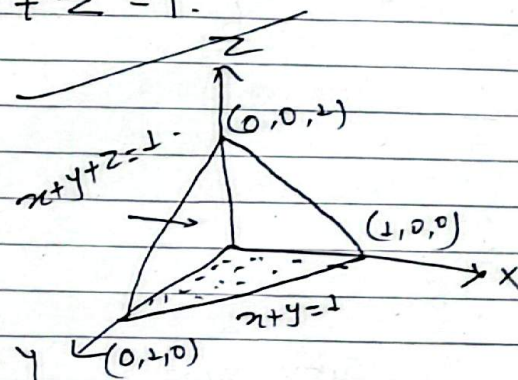
$$= 4\sqrt{2a} \left[2a \frac{y^{3/2}}{3/2} - \frac{y^{5/2}}{5/2} \right]_a^{2a}$$

=

Volume by triple integral:

Find the volume of the tetrahedron enclosed by the axis and the plane $x+y+z=1$.

⇒



$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx$$

$$= \int_0^1 \int_0^{1-x} [z]_0^{1-x-y} dy dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y) dy dx$$

$$= \int_0^1 \left[y - \frac{y^2}{2} - xy \right]_0^{1-x} dx$$

$$\int (a-bx)^n dx = \frac{(a-bx)^{n+1}}{(-b)(n+1)}$$

$$= \int_0^1 \left[1-x \cdot \frac{(1-x)^2}{2} - x(1-x) \right] dx$$

$$= \left[x \cdot \frac{x^2}{2} + \frac{(1-x)^3}{6} - \frac{x^2}{2} + \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{6}$$

$$= \frac{6-3}{18}$$

$$= \frac{1}{6} \text{ Cubic unit.}$$

evaluating the integral by changing cylindrical & spherical coordinate

Case(I) How to change Cartesian to cylindrical polar coordinate.

⇒ We can change Cartesian to cylindrical form as the coordinate.

$$(x, y, z) = (r \cos \theta, r \sin \theta, z), \quad r = \sqrt{x^2 + y^2}$$

$$0 \leq \theta \leq 2\pi$$

$$dv = dx dy dz = r dr d\theta dz$$

⇒ It is used for case of cylindrical condition

Case(II): How to change Cartesian to spherical polar coordinate.

$$(x, y, z) = (r, \theta, \phi)$$

$$\Rightarrow (x, y, z) = (r \cos \phi \cos \theta, r \cos \phi \sin \theta, r \sin \phi)$$

$$0 \leq \theta \leq 2\pi, \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

$$\text{or} \\ 0 < \phi < \pi$$

$$dV = r^2 \sin \phi \, dr \, d\phi \, d\theta$$

it is used for the case of spherical condition.

$$r^2 = x^2 + y^2 + z^2$$

For example:

Find the volume of the sphere.
 $x^2 + y^2 + z^2 = a^2$ under the condition.

$$\iiint_V \frac{1}{x^2 + y^2 + z^2} \, dx \, dy \, dz$$

$$\Rightarrow \text{Since } x = r \cos \phi \cos \theta, \quad y = r \cos \phi \sin \theta$$

$$z = r \sin \phi, \quad x^2 + y^2 + z^2 = r^2$$

$$dV = dx \, dy \, dz = r^2 \sin \phi \, dr \, d\phi \, d\theta$$

$$\text{Now, } V = \int_0^\pi \int_0^{2\pi} \int_0^a \frac{1}{r^2} \cdot r^2 \sin \phi \, dr \, d\phi \, d\theta$$

$$= \int_0^\pi \int_0^{2\pi} \sin \phi \left[r \right]_0^a \, d\phi \, d\theta$$

$$= \int_0^\pi \int_0^{2\pi} \sin \phi \cdot a \, d\phi \, d\theta$$

$$= \int_0^\pi a \sin \phi \left[\theta \right]_0^{2\pi} \, d\phi$$

$$= \int_0^\pi a \sin \phi \cdot (2\pi - 0) \, d\phi$$

$$= 2\pi \cdot a \int_0^\pi \sin \phi \, d\phi$$

$$= 2\pi \cdot a \left[-\cos \phi \right]_0^\pi$$

$$= 2\pi \cdot a \cdot [-\cos \pi + \cos 0]$$

$$= 2\pi a [1 + 1]$$

$$= 4\pi a \text{ cubic unit.}$$

$dA = dydz$
 $r d\theta dr$

Triple integration by using cylindrical and spherical polar condition.

1. Cylindrical polar coordinate:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}, \quad 0 \leq \theta \leq 2\pi$$

$$dv = dx dy dz = r dr d\theta dz$$

$$\Rightarrow (x, y, z) = (r \cos \theta, r \sin \theta, z)$$

2. Spherical polar coordinate:

$$x = r \cos \phi \cos \theta, \quad y = r \cos \phi \sin \theta, \quad z = r \sin \phi$$

$$0 \leq \theta \leq 2\pi, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2} \quad \text{or} \quad 0 < \phi < \pi$$

$$dv = dx dy dz = r^2 \sin \phi dr d\theta d\phi$$

Note: If question $x^2 + y^2$ then use cylindrical polar coordinate.

If question $x^2 + y^2 + z^2$ then use spherical polar coordinate.

Ex. Evaluate the integral $\iiint (x^2 + y^2 + z^2) dx dy dz$ over the volume of sphere $x^2 + y^2 + z^2 = 1$.

$\Rightarrow$ Since by using spherical coordinate:

$$x = r \cos \phi \cos \theta$$

$$y = r \cos \phi \sin \theta$$

$$z = r \sin \phi$$

$$dv = dx dy dz = r^2 \sin \phi dr d\theta d\phi$$

and since the given sphere $x^2 + y^2 + z^2 = 1$.

$$0 \leq \theta \leq \pi, \quad 0 < \phi < \frac{\pi}{2}, \quad 0 \leq r \leq 1$$

$$V = \int_0^{\pi} \int_0^{\pi/2} \int_0^1 r^2 \cdot r^2 \sin \phi dr d\theta d\phi$$

$$= \int_0^{\pi} \int_0^{\pi/2} \left[\frac{r^5}{5} \right]_0^1 \sin \phi d\theta d\phi$$

$$= \int_0^{\pi} \int_0^{\pi/2} \frac{1}{5} \sin \phi d\theta d\phi$$

$$= 8 \int_0^{\pi/2} \frac{1}{5} \sin \phi \left[0 \right]_0^{\pi/2} d\phi$$

$$= 8 \int_0^{\pi/2} \frac{1}{5} \sin \phi \left[\frac{\pi}{2} - 0 \right] d\phi$$

$$= \frac{8\pi}{20} \int_0^{\pi/2} \sin \phi d\phi$$

$$= \frac{8\pi}{20} \left[-\cos \phi \right]_0^{\pi/2}$$

$$= \frac{8\pi}{20} \left[-\cos \frac{\pi}{2} + \cos 0 \right]$$

$$= \frac{4\pi}{5} \text{ Cubic unit.}$$

• Note: Integration by using Dirichlet theorem / condition.

if V is a region bounded by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x+y+z \leq 1$ ($\therefore$ in first octant) then.

$$\int_V \int \int x^{\alpha-1} y^{\beta-1} z^{\gamma-1} dx dy dz = \frac{1}{\sqrt{\alpha+\beta+\gamma+1}}$$

$$f(t) * g(t) = \int_0^t F(t) \cdot g(t) dt$$

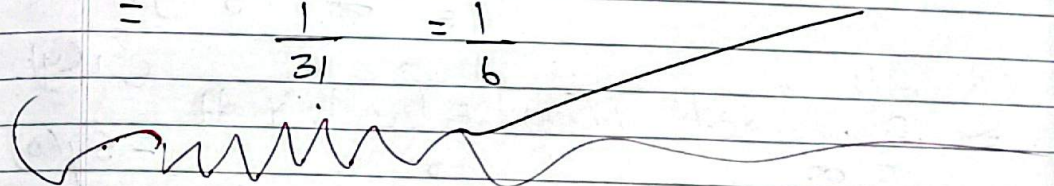
Ex. Find the volume of an object bounded by $x, y, z \geq 0$ and $x+y+z=1$.

$$\Rightarrow V = \int \int \int x^{\alpha-1} y^{\beta-1} z^{\gamma-1} dx dy dz$$

$$= \int \int \int x^{1-1} y^{1-1} z^{1-1} dx dy dz$$

$$= \frac{1}{1+1+1+1}$$

$$= \frac{1}{3!} = \frac{1}{6}$$



h.w Compute: $\int \int \int x y z dx dy dz$ over

$$x \geq 0, y \geq 0, z \geq 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\left(\frac{a^2 b^2 c^2}{2520} \right)$$

Exam-1 term

Question Practice:

unit-3 Laplace transform:

Example $f(t) = 1 = t^0$

By the definition,

$$\begin{aligned} L[f(t)] &= L[1] = \int_0^{\infty} e^{-st} (1) dt \\ &= \int_0^{\infty} e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_0^{\infty} \\ &= \left[\frac{e^{-\infty}}{-s} - \left(\frac{e^0}{-s} \right) \right] \\ &= \frac{1}{s} \# \end{aligned}$$

Ex. $f(t) = t$

$$\begin{aligned} \Rightarrow \text{By the def}^n \quad L[f(t)] &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^{\infty} e^{-st} t \cdot dt \\ &= \left[t \frac{e^{-st}}{-s} - \int \left(\frac{dt}{dt} \int e^{-st} dt \right) dt \right]_0^{\infty} \\ &= \left[t \frac{e^{-st}}{-s} - \int \frac{e^{-st}}{-s} dt \right]_0^{\infty} \\ &= \left[t \frac{e^{-st}}{-s} + \frac{1}{s} \int e^{-st} dt \right]_0^{\infty} \end{aligned}$$

(1)

$$\begin{aligned} &= \left[t \frac{e^{-st}}{-s} - \frac{1}{s^2} e^{-st} \right]_0^{\infty} \\ &= 0 - 0 - 0 + \frac{1}{s^2} e^0 \\ &= \frac{1}{s^2} \# \end{aligned}$$

Similarly,

$$\begin{aligned} &= \left[t \frac{e^{-st}}{-s} - \frac{1}{s^2} e^{-st} \right]_0^{\infty} \\ &= \left[0 - 0 - 0 + \frac{1}{s^2} \right] = \frac{1}{s^2} \end{aligned}$$

Ex. $f(t) = e^{at}, s > a > 0$

$$\begin{aligned} L[f(t)] &= \int_0^{\infty} e^{-st} e^{at} dt \\ &= \int_0^{\infty} e^{-t(s-a)} dt = \left[\frac{e^{-t(s-a)}}{-(s-a)} \right]_0^{\infty} \\ &= 0 - \left(\frac{1}{-(s-a)} \right) = \frac{1}{s-a} \end{aligned}$$

Example: $f(t) = \cos \omega t$

$$L[\cos \omega t] = \int_0^{\infty} e^{-st} \cos \omega t \cdot dt$$

$$= \left[\cos \omega t \frac{e^{-st}}{-s} - \int \left[-\sin \omega t \frac{e^{-st}}{-s} \right] dt \right]_0^{\infty}$$

$$= \left[\cos \omega t \frac{e^{-st}}{-s} - \frac{\sin \omega t e^{-st}}{s} \right]_0^{\infty}$$

Properties:

Linear property:

$$\Rightarrow \mathcal{L}[af(t) + bg(t)] = a \mathcal{L}[f(t)] + b \mathcal{L}[g(t)].$$

Proof. By.

$$\mathcal{L}[af(t) + bg(t)] = \int_0^{\infty} e^{-st} [af(t) + bg(t)] dt$$

$$= a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt$$

$$= a \mathcal{L}[f(t)] + b \mathcal{L}[g(t)]. \text{ proved}$$

2003 Fall:

$$\int_0^{\log 2} \int_0^2 \frac{1}{e^y} dx dy$$

$$\Rightarrow \int_0^{\log 2} \int_0^2 \frac{1}{e^y} dx dy = \int_0^{\log 2} (2 - e^y) dy = [2y - e^y]_0^{\log 2}$$

$$= 2 \log 2 - e^{\log 2} - (0 - e^0)$$

$$= 2 \log 2 - 2 + 1 = 2 \log 2 - 1$$

$$\textcircled{a} \int_0^3 \int_0^2 (4-y)^2 dy dx$$

$$\Rightarrow \int_0^3 \int_0^2 (16 - 8y + y^2) dy dx$$

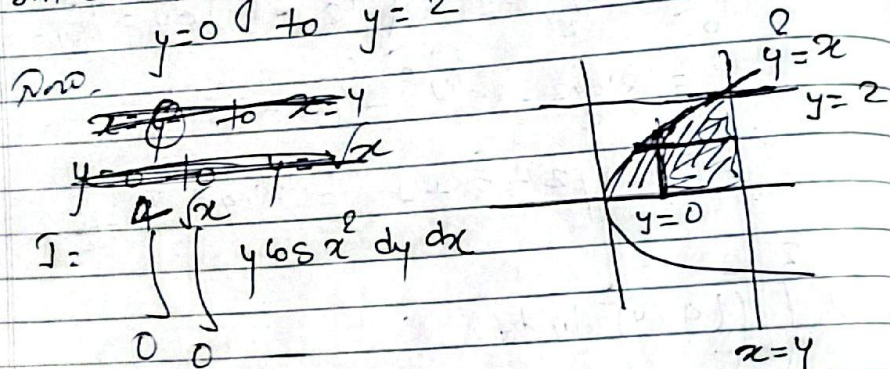
$$= \int_0^3 \left[16y - 4y^2 + \frac{y^3}{3} \right]_0^2 dx$$

$$= \int_0^3 \left(32 - 16 + \frac{8}{3} \right) dx$$

$$= \frac{56}{3} \left[x \right]_0^3 = \frac{56}{3} [3 - 0] = 56$$

Example: 4]
$$\int_0^2 \int_{y^2}^4 y \cos x^2 dx dy$$

Since. $x = y^2$ to $x = 4$
 $y = 0$ to $y = 2$



Now.
$$I = \int_0^4 \int_0^{\sqrt{x}} y \cos x^2 dy dx$$

$$= \int_0^4 \cos x^2 \left[\frac{y^2}{2} \right]_0^{\sqrt{x}} dx$$

$$I = \int_0^4 \cos x^2 \cdot \frac{\sqrt{x}}{2} dx$$

Put $x = z$
 $2x = dz$

$z = 0, z = 16$

Now.
$$dx = \frac{dz}{2x}$$

$$\int_0^{16} \cos z \cdot \frac{\sqrt{z}}{2} \cdot \frac{dz}{2\sqrt{z}}$$

Now.
$$\int_0^{16} \cos z \cdot \frac{1}{2} dz$$

$$= \frac{1}{2} [\sin z]_0^{16}$$

$$= \frac{1}{2} (\sin 16 - \sin 0)$$

$$= \frac{\sin 16}{2}$$

(6)
$$\int_0^{\sqrt{2}} \int_{\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y dx dy$$

$\Rightarrow$ Since $x = \sqrt{4-2y^2} \Rightarrow x^2 = 4-2y^2$

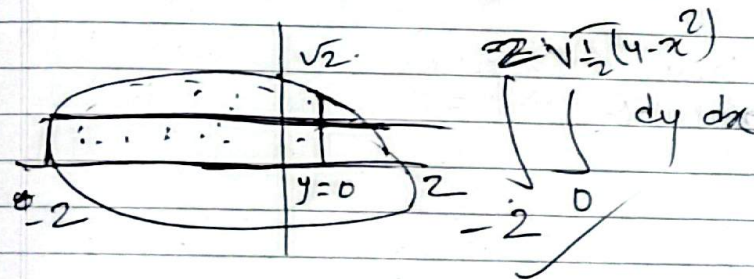
$x = \sqrt{4-2y^2} \Rightarrow x^2 = 4-2y^2$

$y = 0$ to $y = \sqrt{2}$

Since. $x^2 = 4-2y^2$

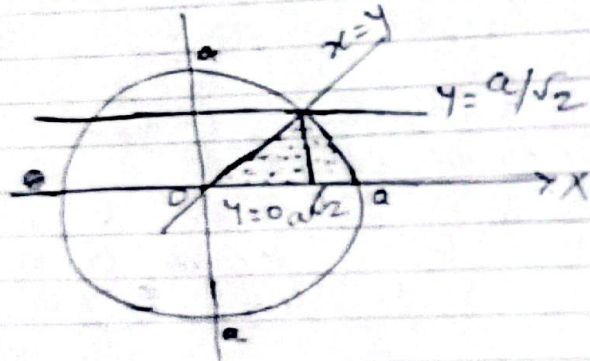
$x^2 + 2y^2 = 4$

$\frac{x^2}{4} + \frac{y^2}{2} = 1$ which is an ellipse with centre (0,0) and vertices (+2, 0) and (0, $\sqrt{2}$)



$$\textcircled{d} \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2) \, dx \, dy$$

$\Rightarrow$ Since $x=y$ to $x=\sqrt{a^2-y^2} \Rightarrow x^2+y^2=a^2$
 $y=0$ to $y=\frac{a}{\sqrt{2}}$



By Polar Condition.
 $x = r \cos \theta \Rightarrow r = 0$

Differentiation and Integration.

Theory-1: of vector funcⁿ:

The necessary & sufficient condⁿ for a vector $\vec{a}$ of the scalar variable t to have constant magnitude is $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$.

⇒ Proof: Suppose that $\vec{a}$ (funcⁿ) has constant magnitude.

We have to prove that

$$\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$$

$$\text{let } \vec{a} \cdot \vec{a} = |\vec{a}|^2$$

Differentiating with respect to t ;

$$\Rightarrow \frac{d}{dt}(\vec{a} \cdot \vec{a}) = \frac{d}{dt}[|\vec{a}|^2]$$

$$\Rightarrow \vec{a} \cdot \frac{d\vec{a}}{dt} + \vec{a} \cdot \frac{d\vec{a}}{dt} = 0$$

$$2 \vec{a} \cdot \frac{d\vec{a}}{dt} = 0$$

$$\therefore \vec{a} \cdot \frac{d\vec{a}}{dt} = 0$$

proved.

Similarly, Suppose

$\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$, then we have to show that $\vec{a}$ has a constant magnitude.

$$\text{We have. } \vec{a} \cdot \vec{a} = |\vec{a}|^2$$

Differentiate on both side with r. to 't'.

$$\frac{d}{dt}(\vec{a} \cdot \vec{a}) = \frac{d}{dt}|\vec{a}|^2$$

$$\vec{a} \cdot \frac{d\vec{a}}{dt} + \vec{a} \cdot \frac{d\vec{a}}{dt} = 2|\vec{a}| \frac{d}{dt}|\vec{a}|$$

$$(2 \vec{a} \cdot \frac{d\vec{a}}{dt}) = 2|\vec{a}| \frac{d}{dt}|\vec{a}|$$

$$0 = |\vec{a}| \frac{d}{dt}|\vec{a}| \quad \left[\because \vec{a} \cdot \frac{d\vec{a}}{dt} = 0 \right]$$

Since $\vec{a} \neq 0$ so, $|\vec{a}| \neq 0$ therefore

$$|\vec{a}| \frac{d}{dt}|\vec{a}| = 0$$

$$\Rightarrow \frac{d}{dt}|\vec{a}| = 0 \quad \text{proved.}$$

C.W

1. If $\vec{r}_1 = t^2 \vec{i} - t \vec{j} + (2t+1) \vec{k}$, $\vec{r}_2 = (2t-3) \vec{i} + \vec{j} - t \vec{k}$

Find:

(a) $\frac{d}{dt} (\vec{r}_1 \cdot \vec{r}_2)$ at $t=1$. (b) $\frac{d}{dt} (\vec{r}_1 \times \vec{r}_2)$ at $t=1$.

Q.2] If $\vec{r} = \vec{a} e^{nt} + \vec{b} e^{-nt}$ where $\vec{a}$ & $\vec{b}$ are constant vector. Show that: $\frac{d^2 \vec{r}}{dt^2} = n^2 \vec{r}$

a. $\Rightarrow \vec{r}_1 \cdot \vec{r}_2 = (t^2 \vec{i} - t \vec{j} + (2t+1) \vec{k}) \cdot ((2t-3) \vec{i} + \vec{j} - t \vec{k})$

$$= [t^2(2t-3)] - t - t(2t+1)$$

$$= 2t^3 - 3t^2 - t - 2t^2 - t$$

$$= 2t^3 - 5t^2 - 2t$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^2 & -t & 2t+1 \\ 2t-3 & 1 & -t \end{vmatrix}$$

$$= (t^2 - 2t - 1) \vec{i} + (-t^3 - 4t^2 - 2t + 6t + 3) \vec{j} + (t^2 + 2t^2 - 3t) \vec{k}$$

$$= (t^2 - 2t - 1) \vec{i} - (t^3 + 4t^2 + 2t - 6t - 3) \vec{j} + (3t^2 - 3t) \vec{k}$$

$$= (t^2 - 2t - 1) \vec{i} + (-t^3 - 4t^2 - 4t - 3) \vec{j} + (3t^2 - 3t) \vec{k}$$

Now.

(a) $\frac{d}{dt} (\vec{r}_1 \cdot \vec{r}_2) = \frac{d}{dt} (2t^3 - 5t^2 - 2t)$

$$= 6t^2 - 10t - 2$$

at $t=1$.

$$\frac{d}{dt} (\vec{r}_1 \cdot \vec{r}_2) = 6 - 10 - 2 = -6$$

(b) $\frac{d}{dt} (\vec{r}_1 \times \vec{r}_2) = \frac{d}{dt} (t^2 - 2t - 1, -t^3 - 4t^2 - 4t - 3, 3t^2 - 3t)$

$$= 2t - 2, -3t^2 - 8t - 4, 6t - 3$$

at $t=1$.

$$\frac{d}{dt} (\vec{r}_1 \times \vec{r}_2) = (2 - 2, -3 - 8, 6 - 3) = (0, -11, 3)$$

S.P.2. Taking ~~the~~

$$\vec{r} = \vec{a}e^{nt} + \vec{b}e^{-nt}.$$

Differentiate.

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d}{dt} (\vec{a}e^{nt} + \vec{b}e^{-nt}) \\ &= \vec{a} \frac{d}{dt} (e^{nt}) + \vec{b} \frac{d}{dt} (e^{-nt}) \\ &= \vec{a} n e^{nt} + \vec{b} (-n) e^{-nt}.\end{aligned}$$

Some imp formula :-

1. Unit tangent vector

$$\hat{T} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|}$$

2. Velocity ($\vec{v}$) = $\frac{d\vec{r}}{dt}$

3. acceleration ($\vec{a}$) = $\frac{d^2\vec{r}}{dt^2}$

4. Component of Velocity = $\vec{v} \cdot \hat{n}$

5. comp. of acceleration = $\vec{a} \cdot \hat{n}$

6. Displacement = $\vec{r}$

Find unit tangent vector where, $x = 3\cos t$
 $y = 3\sin t$, $z = 4t$

$$\Rightarrow \text{We have, } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = 3\cos t\vec{i} + 3\sin t\vec{j} + 4t\vec{k}$$

For the unit vector,

Differentiate w.r. to t , we get.

$$\frac{d\vec{r}}{dt} = -3\sin t\vec{i} + 3\cos t\vec{j} + 4\vec{k}$$

$$\begin{aligned} \text{Now, tangential unit vector} &= \frac{\frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|} = \frac{-3\sin t\vec{i} + 3\cos t\vec{j} + 4\vec{k}}{\sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}} \\ &= \frac{-3\sin t\vec{i} + 3\cos t\vec{j} + 4\vec{k}}{\sqrt{25}} \\ &= \frac{-3\sin t\vec{i} + 3\cos t\vec{j} + 4\vec{k}}{5} \end{aligned}$$

Q. A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is time. Determine its velocity and acceleration vectors and also the magnitude of velocity and acceleration.

$$\Rightarrow \text{Since, } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = e^{-t}\vec{i} + 2\cos 3t\vec{j} + 2\sin 3t\vec{k}$$

Diff. w.r. to t we get.

$$\frac{d\vec{r}}{dt} = e^{-t}\vec{i} + (-6\sin 3t)\vec{j} + 6\cos 3t\vec{k}$$

$$\therefore \text{Velocity } (\vec{v}) = \frac{d\vec{r}}{dt} = e^{-t}\vec{i} - 6\sin 3t\vec{j} + 6\cos 3t\vec{k}$$

at $t=0 \Rightarrow$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{i} + 6\vec{k}$$

Also, acceleration, again differentiate $\frac{d\vec{r}}{dt}$ w.r. to t .

$$\frac{d^2\vec{r}}{dt^2} = e^{-t}\vec{i} - 18\cos 3t\vec{j} - 18\sin 3t\vec{k}$$

$$\text{So, for acc. at } t=0, \Rightarrow \vec{a} = \frac{d^2\vec{r}}{dt^2} = \vec{i} - 18\vec{j}$$

$$\begin{aligned} \text{Now, } |\vec{a}| &= \sqrt{1^2 + (-18)^2} \\ &= \sqrt{1 + 324} \end{aligned}$$

$$= \sqrt{325} = 5\sqrt{13}$$

$$\text{Again, } |\vec{v}| = \sqrt{1^2 + 6^2} = \sqrt{1 + 36} = \sqrt{37}$$

Q. Find the Component of Velocity and acceleration of a moving particle $x = t^2 + 1$, $y = t^2$, $z = 2t + 5$ at $t = 1$. in the direction of $\vec{i} + \vec{j} + 3\vec{k}$

⇒ Since, $\vec{r} = (t^2 + 1)\vec{i} + t^2\vec{j} + (2t + 5)\vec{k}$ ①
Diff. w.r to 't'. we get.

$$\vec{v} = \frac{d\vec{r}}{dt} = 2t\vec{i} + 2t\vec{j} + 2\vec{k}$$

at $t = 1$, $\Rightarrow \vec{v} = 2\vec{i} + 2\vec{j} + 2\vec{k}$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 2\vec{i} + 2\vec{j}$$

at $t = 1 \Rightarrow \vec{a} = 2\vec{i} + 2\vec{j}$

Component of Velocity = $\vec{v} \cdot \hat{n} = (2\vec{i} + 2\vec{j} + 2\vec{k}) \cdot \hat{n}$

$$= (2\vec{i} + 2\vec{j} + 2\vec{k}) \cdot \frac{\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{1^2 + 1^2 + 3^2}}$$

$$= \frac{2 + 2 + 6}{\sqrt{11}} = \frac{10}{\sqrt{11}} = \sqrt{11}$$

Again.

Comp. of acceleration = $\vec{a} \cdot \hat{n} = (2\vec{i} + 2\vec{j} + 0\vec{k}) \cdot \hat{n}$

$$= (2\vec{i} + 2\vec{j} + 0\vec{k}) \cdot \frac{\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{1^2 + 1^2 + 3^2}}$$

$$= \frac{2 + 2 + 0}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

1.8. Q. If the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$ show that $[\vec{r} \vec{r}' \vec{r}'] = 36$ at $t = 1$.

Q. Let $\vec{a} = 12\cos 2t\vec{i} - 8\cos 2t\vec{j} + 16\vec{k}$ be the acceleration particle at any time t . find the velocity $\vec{v}$ and displacement $\vec{r}$ at any time t and given $\vec{r} = 0$, $\vec{v} = 0$ at $t = 0$.

Exactness condition on vector function.

Let $f_1 dx + f_2 dy + f_3 dz$ be the function of three variable. Where f_1, f_2 and f_3 are continuous in first order partial derivatives. Let $\int f_1 dx + f_2 dy + f_3 dz$ is integrable then $f_1 dx + f_2 dy + f_3 dz$ must satisfy.

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} ; \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} ; \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}$$

⇒ The required solution is.

$$\int f_1 dx + \int (\text{term of } f_2 \text{ free from } x) dy + \int (\text{term of } f_3 \text{ free from } x \text{ and } y) dz + C$$

Q. Show that the given integral is exact and evaluate the integral.

$$\int_{(1,0,2)}^{(-2,1,3)} (6xy^3 + 2z^2) dx + 9x^2y^2 dy + (4xz+1) dz$$

$$\Rightarrow \text{here, } f_1 = 6xy^3 + 2z^2, f_2 = 9x^2y^2, f_3 = 4xz+1$$

$$\text{For, } \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \Rightarrow \frac{\partial (6xy^3 + 2z^2)}{\partial y} = \frac{\partial (9x^2y^2)}{\partial x} \\ \Rightarrow 18xy^2 = 18xy^2.$$

$$\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \Rightarrow \frac{\partial (9x^2y^2)}{\partial z} = \frac{\partial (4xz+1)}{\partial y}$$

$$\Rightarrow 0 = 0$$

And.

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} \Rightarrow \frac{\partial (6xy^3 + 2z^2)}{\partial z} = \frac{\partial (4xz+1)}{\partial x}$$

$$\Rightarrow 4z = 4z.$$

So, given integral is exact.

The required soln:-

$$\int f_1 dx + \int (\text{term of } f_2 \text{ free from } x) dy + \int (\text{term of } f_3 \text{ free from } x \text{ and } y) dz \\ \Rightarrow \int_{(1,0,2)}^{(-2,1,3)} (6xy^3 + 2z^2) dx + \int_0^1 0 dy + \int_2^3 1 dz$$

$$\Rightarrow \left[\frac{6x^2y}{2} + \frac{2z^2x}{2} \right]_{(1,0,2)}^{(-2,4,3)} + \left[x \right]_{(1,0,2)}^{(-2,1,3)}$$

$$\Rightarrow \left[3(4-1) + (9-4)(-2-1) \right] + (3-2)$$

$$= \left[-3 \times 3 + 2(5)(-3) \right] + 1$$

$$= -9 + (-30) + 1$$

=

$$\frac{dw}{dx} (3, 2)$$

$$Q.1. \int_{(0,1,1/2)}^{(3,2)} [y^2 \cos x dx + (2y \sin x + e^{2z}) dy + 2ye^{2z} dz]$$

$$\Rightarrow \text{Here } f_1 = y^2 \cos x, f_2 = 2y \sin x + e^{2z}, f_3 = 2ye^{2z}$$

$$\text{For } \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \Rightarrow \frac{\partial (y^2 \cos x)}{\partial y} = \frac{\partial (2y \sin x + e^{2z})}{\partial x}$$

$$\Rightarrow 2y \cos x = 2y \cos x$$

$$\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \Rightarrow \frac{\partial (2y \sin x + e^{2z})}{\partial z} = \frac{\partial (2ye^{2z})}{\partial y}$$

$$\Rightarrow 2e^{2z} = 2e^{2z}$$

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} \Rightarrow \frac{\partial (y^2 \cos x)}{\partial z} = \frac{\partial (2ye^{2z})}{\partial x}$$

$$= 0 = 0$$

So, given integral is exact.
Now,

$$\int_{(0,1,1/2)}^{(1/2,3,2)} y^2 \cos x \, dx + \int_{(0,1,1/2)}^{(1/2,3,2)} e^{2z} \, dy + \int_{(0,1,1/2)}^{(1/2,3,2)} 0 \, dz$$

$$\Rightarrow \left[y^2 \sin x \right]_{(0,1,1/2)}^{(1/2,3,2)} + \left[e^{2z} y \right]_{(0,1,1/2)}^{(1/2,3,2)}$$

$$\Rightarrow \left(\left(\frac{3^2}{2} - 1 \right) (\sin \frac{\pi}{2} - \sin 0) \right) + \left(e^{2 \cdot 3} - e^{2 \cdot 1/2} \right) (3 - 1)$$

$$= (9 - 1)(1 - 0) + (e^4 - e^1) \cdot 2$$

$$= 8 + 2(e^4 - e)$$

$$\Rightarrow \left[9 \cdot \frac{\sin \pi}{2} - 1 \cdot \sin 0 \right] + \left[e^{2 \cdot 3} \cdot 3 - e^{2 \cdot 1/2} \cdot 1 \right]$$

$$\Rightarrow 9 + 3e^4 - e$$

#

Vector field and vector differential operator.

$$\nabla = \text{Nabla} = \text{nabla} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right)$$

1. Gradient of Scalar Function.

Let $\phi(x, y, z)$ be a scalar function of three variables x, y, z . The gradient of ϕ is defined as $\text{grad } \phi$ or $\nabla \phi$.

$$\text{i.e. } \text{grad } \phi = \nabla \phi = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \phi$$

$$= \left(\vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \right)$$

Note:

1. Gradient of ϕ i.e. $\text{grad } \phi$ is a normal vector containing to the surface ϕ and whose magnitude is max^m rate of change of ϕ along this normal.

2. If ϕ is given constant function then $\text{grad } \phi = 0$.

Q.1 Find grad ϕ of e^{xyz} .

$\Rightarrow$ Here, $\phi = e^{xyz}$

then.

$$\text{grad } \phi = \left(\vec{i} \frac{\partial e^{xyz}}{\partial x} + \vec{j} \frac{\partial e^{xyz}}{\partial y} + \vec{k} \frac{\partial e^{xyz}}{\partial z} \right)$$

$$= yze^{xyz} \vec{i} + xze^{xyz} \vec{j} + xy e^{xyz} \vec{k}$$

Q.2 Find curl of a scalar function $\phi(x, y, z)$.

$$\text{curl } \phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix}$$

Find curl $\vec{F}$, $\vec{F} = xyz\vec{i} + yz\vec{j} + zx\vec{k}$.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{0}$$

$$= \left(\frac{\partial zx}{\partial y} - \frac{\partial yz}{\partial z} \right) \vec{i} + \vec{j}$$

Directional derivatives.
The directional derivative of a function f in the direction $\vec{a}$ is denoted by $D_{\vec{a}} f$ and defined by $D_{\vec{a}} f = \text{grad} f \cdot \vec{a}$
 $= \text{grad} f \cdot \frac{\vec{a}}{|\vec{a}|}$

Example: Find the directional derivative of $f = x^2 + y^2$ at $P(2, 1)$ in the direction $\vec{a} = 2\vec{i} - 4\vec{j}$.

⇒ Solution: $f = x^2 + y^2$, $\vec{a} = 2\vec{i} - 4\vec{j}$
for $\text{grad} f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y}$
 $= \vec{i} \frac{\partial (x^2 + y^2)}{\partial x} + \vec{j} \frac{\partial (x^2 + y^2)}{\partial y}$
 $= 2x\vec{i} + 2y\vec{j} = 2\vec{i} + 2\vec{j}$ at $P(2, 1)$

Again, $|\vec{a}| = \sqrt{2^2 + (-4)^2} = \sqrt{20}$

Now, $D_{\vec{a}} f = \frac{(2\vec{i} + 2\vec{j}) \cdot (2\vec{i} - 4\vec{j})}{\sqrt{20}}$
 $= \frac{4 - 8}{\sqrt{20}} = \frac{-4}{\sqrt{20}}$ #

Q. Find the directional derivative of $f = xy^2 + yz^3$ at $P(2, -1, 1)$ along the direction of the Normal to the surface: $x \log z - y^2 + 4 = 0$ at $(-1, 2, 1)$.

⇒ Here, $f = xy^2 + yz^3$
 $\text{grad} f = \vec{i} \frac{\partial (xy^2 + yz^3)}{\partial x} + \vec{j} \frac{\partial (xy^2 + yz^3)}{\partial y} + \vec{k} \frac{\partial (xy^2 + yz^3)}{\partial z}$
 $= y^2 \vec{i} + (2xy + z^3) \vec{j} + (3yz^2) \vec{k}$
 $= \vec{i} + (2\vec{j} + 1\vec{k})$ at $P(2, -1, 1)$

$\text{grad} \phi = \nabla \phi = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \phi$
 $= \left(\vec{i} \frac{\partial (x \log z - y^2 + 4)}{\partial x} + \vec{j} \frac{\partial (x \log z - y^2 + 4)}{\partial y} + \vec{k} \frac{\partial (x \log z - y^2 + 4)}{\partial z} \right)$
 $= \vec{i} \log z + (-2y) \vec{j} + \frac{x}{z} \vec{k}$
 $= \vec{i} \cdot 0 - 4\vec{j} + (-1) \vec{k}$ at $(-1, 2, 1)$
 $(\vec{a}) = -4\vec{j} - \vec{k}$

Now, $\nabla \cdot \vec{a} = \text{grad} f \cdot \hat{a}$

$$= (\vec{i} + 3\vec{j} - \vec{k}) \cdot \frac{(-4\vec{j} - \vec{k})}{\sqrt{(-4)^2 + (-1)^2}}$$

$$= \frac{-4 + 3}{\sqrt{17}} = \frac{-1}{\sqrt{17}}$$

$$\vec{a} = \vec{a} \cdot \hat{a} \quad (\hat{a} \times \hat{a})$$

Unit - 4

Diff

Conservative field or Conservative function:
A force $\vec{F}$ is said to be conservative field if it is derivable from the potential funcⁿ. ϕ
or, $\vec{F} = \text{grad } \phi$ i.e. $\vec{F} = \nabla \phi \Rightarrow \vec{F} = \text{grad } \phi$
= $\nabla \phi$

[To show that conservative field $\text{curl } \vec{F} = 0$]

Solenoidal vector: $\nabla \cdot \vec{F} = 0$

(v.v.)

Q. Show the vector $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ is conservative and find scalar potential funcⁿ ϕ s.t. $\vec{F} = \nabla \phi$.

$\Rightarrow$ since, $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$

we have to show that, $\vec{F}$ is conservative for,

$$\text{curl } \vec{F} = 0 \Rightarrow \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - yz) & (y^2 - zx) & (z^2 - xy) \end{vmatrix}$$

$$\Rightarrow \vec{i}(-y - (-x)) + \vec{j}(-y - (-y)) + \vec{k}(-z - (-z))$$

$$\Rightarrow \vec{i}(-y + x) + \vec{j}(0) + \vec{k}(0)$$

let ϕ be a scalar funcⁿ we know.

$$\vec{F} = \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

multiplying both sides by $d\vec{r}$.

$$\vec{F} \cdot d\vec{r} = \left(\vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$= d\phi \left[\because \frac{dy}{dx} = \frac{\partial v}{\partial x} \frac{dx}{dy} + \frac{\partial v}{\partial y} \frac{dy}{dz} + \frac{\partial v}{\partial z} \frac{dz}{dz} \right]$$

Now,

$$\Rightarrow d\phi = \vec{F} \cdot d\vec{r}$$

Taking integration on both side,

$$\Rightarrow \int d\phi = \int \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \phi = \int [(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}] \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int [(x^2 - yz)dx + (y^2 - zx)dy + (z^2 - xy)dz]$$

Date _____
Page _____

$$= \int f_1 dx + \int (\text{term w/ } f_2 \text{ free from } x) dy + \int (\text{term w/ } f_3 \text{ free from } x \text{ and } y) dz$$

$$= \int (x^2 - yz) dx + \int y^2 dy + \int z^2 dz$$

$$= \frac{x^3}{3} - yzx + \frac{y^3}{3} + \frac{z^3}{3} + C \quad \underline{\underline{\text{Ans}}}$$

unit - 5

Date _____
Page _____

Line Integrals & Surface Integrals.

- ① Line integral
- ② Surface integral [Green's theorem]
- ③ Volume integral
- ④ Stokes theorem.

Work done $\Rightarrow \vec{F} \cdot d\vec{r}$

$$\frac{d\vec{r}}{dt} = \vec{v}$$

$$d\vec{r} = \vec{v} dt$$

Surface Integral (8 marks)

Note:

$$\iint_S \vec{F} \cdot \vec{n} dA$$

$$\vec{r} = (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v}$$

$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

Q. Evaluate: $\iint_S \vec{F} \cdot \vec{n} dA$ if $\vec{F} = (18z, 12, 3y)$
 S is a surface of the plane $2x + 3y + 6z = 12$
 in first octant.

Given: $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ and
 $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Setting: $x = u, y = v \Rightarrow z = \frac{12 - 2u - 3v}{6}$

then $\vec{r} = u\vec{i} + v\vec{j} + \left(\frac{12 - 2u - 3v}{6}\right)\vec{k}$

For $\vec{n}$: $\vec{r}_u = \vec{i} - \frac{1}{3}\vec{k}$

$\vec{r}_v = \vec{j} - \frac{1}{2}\vec{k}$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1/3 \\ 0 & 1 & -1/2 \end{vmatrix}$$

$$= \vec{i} \left(0 - \left(-\frac{1}{6}\right) \right) + \vec{j} \left(-\frac{1}{2} - 0 \right) + \vec{k} \left(1 - 0 \right)$$

$$= \frac{1}{6}\vec{i} - \frac{1}{2}\vec{j} + \vec{k}$$

and $\vec{F} = (12 - 2u - 3v)\vec{i} + 3v\vec{k}$

then: $\vec{F} \cdot \vec{n} = \vec{F} \cdot \vec{n} = 12 - 2u - 3v - 6 + 3v$

Now, $\iint_S \vec{F} \cdot \vec{n} dA = \int_0^6 \int_0^{6-2u} (6 - 2u) du dv$

$$= \int_0^6 \left[6u - u^2 \right]_0^{6-2u} du$$

$$= \int_0^6 (6 - 2u)(6 - 2u) du$$

$$= \frac{1}{3} \int_0^6 (72 - 12u - 24u + 4u^2) du$$

$$= \frac{1}{3} [72u - 6u^2 - 12u^2 + \frac{4}{3}u^3]_0^6$$

$$\begin{aligned}
 &= \frac{1}{3} \left[724 - \frac{36u^2}{2} + \frac{4u^3}{3} \right]_0^3 \\
 &= \frac{1}{3} \left[72(6-0) - \frac{36}{2}(6^2-0^2) + \frac{4}{3}(6^3-0^3) \right] \\
 &= \frac{1}{3} [456 - 648 + \frac{864}{3}] \quad [x] \\
 &= 24
 \end{aligned}$$

(12) $\iint_S \vec{F} \cdot \hat{n} \, dA$ for $x\vec{i} + y\vec{j} + z\vec{k}$ and S is the upper half of sphere: $x^2 + y^2 + z^2 = a^2$

$\Rightarrow$ Since, $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$, Surface: $x^2 + y^2 + z^2 = a^2$

Setting, $x=u$, $y=v$.

$$\vec{F} = u\vec{i} + v\vec{j} + (\sqrt{a^2 - u^2 - v^2})\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = u\vec{i} + v\vec{j} + (\sqrt{a^2 - u^2 - v^2})\vec{k}$$

$$\vec{r}_u = \vec{i} - \frac{u}{\sqrt{a^2 - u^2 - v^2}}\vec{k}$$

$$\vec{r}_v = \vec{j} - \frac{v}{\sqrt{a^2 - u^2 - v^2}}\vec{k}$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{u}{\sqrt{a^2 - u^2 - v^2}} \\ 0 & 1 & -\frac{v}{\sqrt{a^2 - u^2 - v^2}} \end{vmatrix}$$

$$= \frac{u}{\sqrt{a^2 - u^2 - v^2}}\vec{i} + \frac{v}{\sqrt{a^2 - u^2 - v^2}}\vec{j} + \vec{k}$$

$$\vec{F} \cdot \vec{N} = \frac{u^2}{\sqrt{a^2 - u^2 - v^2}} + \frac{v^2}{\sqrt{a^2 - u^2 - v^2}} + \sqrt{a^2 - u^2 - v^2}$$

$$= \frac{u^2 + v^2 + a^2 - u^2 - v^2}{\sqrt{a^2 - u^2 - v^2}} = \frac{a^2}{\sqrt{a^2 - u^2 - v^2}}$$

Now,

$$\iint_S \vec{F} \cdot \vec{N} \, dA = \iint \left(\frac{a^2}{\sqrt{a^2 - u^2 - v^2}} \right)$$

S

$$= 4 \int_0^{\pi/2} \int_0^a \frac{a^2}{\sqrt{a^2 - r^2}} r \, dr \, d\theta$$

$$= -\frac{4a^2}{2} \int_0^{\pi/2} \int_0^a \frac{-2r}{\sqrt{a^2-r^2}} dr d\theta$$

$$= -2a^2 \int_0^{\pi/2} \left[\sqrt{a^2-r^2} \right]_0^a d\theta$$

$$= -2a^2 \int_0^{\pi/2} -a d\theta$$

$$= 4a^3 \left[\theta \right]_0^{\pi/2}$$

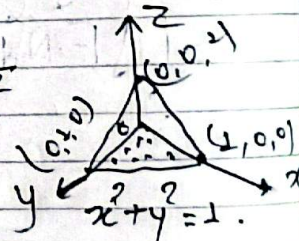
$$= 2\pi a^3$$

Q. 11] Given, $\vec{F} = (yz, zx, xy) = yz\vec{i} + zx\vec{j} + xy\vec{k}$

Surface: $x^2 + y^2 + z^2 = 1$
 $z = \sqrt{1-x^2-y^2}$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Setting.



$$x=4, y=0$$

$$\vec{F} = 0(\sqrt{1-4^2-0^2})\vec{i} + (\sqrt{1-4^2-0^2}) \cdot 4\vec{j} + 4 \cdot 0\vec{k}$$

$$\vec{r} = 4\vec{i} + 0\vec{j} + (\sqrt{1-4^2-0^2})\vec{k}$$

$$\vec{r}_u = \vec{i} - \left(\frac{4}{\sqrt{1-4^2-0^2}} \right) \vec{k}$$

$$\vec{r}_v = \vec{j} - \left(\frac{0}{\sqrt{1-4^2-0^2}} \right) \vec{k}$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{4}{\sqrt{1-4^2-0^2}} \\ 0 & 1 & -\frac{0}{\sqrt{1-4^2-0^2}} \end{vmatrix}$$

$$= \frac{4}{\sqrt{1-4^2-0^2}} \vec{i} + \frac{0}{\sqrt{1-4^2-0^2}} \vec{j} + \vec{k}$$

$$\vec{F} \cdot \vec{N} = 4 \cdot 4 + 0 \cdot 0 + 4 \cdot 1$$

$$= 34$$

Now -

$$\iint_S \vec{F} \cdot \vec{N} = \int_0^1 \int_0^{\sqrt{1-u^2}} 34uv \, dv \, du$$

$$= \int_0^1 34 \left[\frac{v^2}{2} \right]_0^{\sqrt{1-u^2}} du$$

$$= \int_0^1 34 \cdot \frac{1}{2} (1-u^2)^2 du$$

$$= \int_0^1 \frac{34}{2} (1-u^2)^2 du$$

$$= \frac{3}{2} \left[\frac{u^2}{2} - \frac{4u^4}{4} \right]_0^1$$

$$= \frac{3}{2} \left[\frac{1}{2} - 1 \right]$$

$$= \frac{3}{2} \left[-\frac{1}{2} \right] = -\frac{3}{4}$$

$$= \frac{3}{8}$$

(9) Given. $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

Surface of Plane: $2x + 3y + z = 6$

$\therefore z = 6 - 2x - 3y$

Setting, $x = u, y = v$

$\vec{F} = u\vec{i} + v\vec{j} + (6 - 2u - 3v)\vec{k}$

$\vec{r} = u\vec{i} + v\vec{j} + (6 - 2u - 3v)\vec{k}$

$\vec{r}_u = \vec{i} - 2\vec{k}$

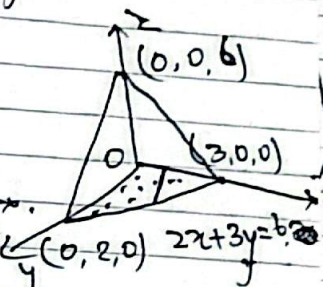
$\vec{r}_v = \vec{j} - 3\vec{k}$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix}$

$$= 2\vec{i} + 3\vec{j} + \vec{k}$$

$\vec{F} \cdot \vec{N} = 2u + 3v + 6 - 2u - 3v$

$$= 6$$



Now $\iint_S \vec{f} \cdot \vec{n} dA = \iint 6 dA.$

$$= 6 \int_0^3 \int_0^{6-\frac{2u}{3}} du dv$$

$$= 6 \int_0^3 \left[\frac{6-2u}{3} \right] du = 2 \left[6u - \frac{2u^2}{2} \right]_0^3 = 2[18-9] = 18$$

$$\begin{aligned}
 &= 2 \left[\frac{2 \cdot 6 \cdot 3}{3} - \frac{2 \cdot 3^2}{2} \right]_0^3 \\
 &= 2[6 \cdot 3 - 9] \\
 &= 2[18-9] \\
 &= 2 \cdot 9 = 18
 \end{aligned}$$

~~$= 6[18-9] = 6 \cdot 9 = 54$
 $= 54$~~

(3) (a) $\oint (2xy^3 dx + 3x^2y^2 dy)$

C: $x^2 + y^2 = 1$

Green's theorem: (statement only):
 let R be the closed region in the xy plane bounded by a curve C .
 let $f_1(x, y)$ & $f_2(x, y)$ be two funcⁿ that are continuous and exists partial derivatives on R . then,

$$\oint_C f_1 dx + \oint_C f_2 dy = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA, \text{ in anticlockwise direction}$$

Note: if vector funcⁿ $\vec{F} = f_1 \vec{i} + f_2 \vec{j}$ is given Green's can be solved as.

$$\iint_S (\text{curl } \vec{F} \cdot \vec{k}) dA$$

Q. Evaluate $\oint 3y dx + \oint 5x dy = \oint (3y dx + 5x dy)$
 Where C is closed path as.
 $0 \leq x \leq 1, 0 \leq y \leq 1$. (Counter clockwise)

⇒ Given,

$$\oint [3y dx + 5x dy], 0 \leq x \leq 1, 0 \leq y \leq 1.$$

Comparing $\oint f_1 dx + f_2 dy$

$$0 < \theta < 2\pi$$

then.

$$f_1 = 3y, f_2 = 5x.$$

$$\frac{\partial f_1}{\partial y} = 3, \frac{\partial f_2}{\partial x} = 5$$

Now,

$$\begin{aligned} \oint_C [3y dx + 5x dy] &= \int_0^1 \int_0^1 (5 - 3) dx dy \\ &= \int_0^1 2 [x]_0^1 dy \\ &= 2 [y]_0^1 \\ &= 2 \end{aligned}$$

$$\oint f_1 dx + f_2 dy = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA.$$

$\vec{F}$

$$\iint_S (\text{curl } \vec{F} \cdot \vec{k}) dA =$$

Q. Evaluate: $\oint_C (x^3 - 3y) dx + (x + \sin y) dy$; C is triangle vertices $(0,0)$, $(1,0)$, $(0,2)$

⇒ Given,
 $\oint_C (x^3 - 3y) dx + (x + \sin y) dy$

Comparing.

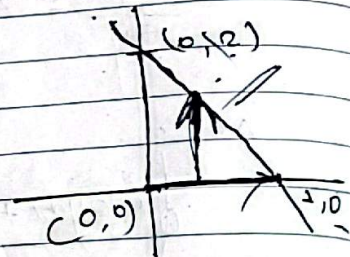
$\oint_C F_1 dx + F_2 dy$ then,

$$F_1 = (x^3 - 3y), \quad F_2 = (x + \sin y)$$

$$\frac{\partial F_1}{\partial y} = -3, \quad \frac{\partial F_2}{\partial x} = 1$$

$$x=0 \text{ to } x=1$$

$$y=0 \text{ to } y=2$$



Now,

$$\oint_C (x^3 - 3y) dx + (x + \sin y) dy = \int_0^1 \int_0^{2-2x} [1 - (-3)] dy dx$$

$$= \int_0^1 4 \left[y \right]_0^{2-2x} dx = 4 \int_0^1 (2-2x) dx$$

$$= 4 \left[2x - x^2 \right]_0^1 = 4(2-1) = 4$$

~~Use~~ Straight line formula:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y - 0 = \frac{2 - 0}{1 - 0} (x - 0)$$

$$y = 2x$$

$$y = 2 - 2x$$

$$y - 0 = \frac{2 - 0}{0 - 1} (x - 0)$$

$$= 4 \int_0^1 (2 - 2x) dx$$

$$= 4 \left[2x - \frac{2x^2}{2} \right]_0^1$$

$$= 4 [2 - 1]$$

$$= 4$$

$$y=0$$

Green's theorem

$$1. \int f_1 dx + f_2 dy = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

$$2. \text{ If } \vec{f} = f_1 \vec{i} + f_2 \vec{j}$$

$$= \iint_S (\text{curl } \vec{f} \cdot \vec{k}) dA$$

③ If the path is closed and known.

$$A = \frac{1}{2} \oint (x dy - y dx)$$

like: circle, astroid, ellipse

Q. Find the area of a circle by using Green's theorem $x^2 + y^2 = a^2$

⇒ We know that.
By Green's theorem on circle, $x^2 + y^2 = a^2$

$$A = \frac{1}{2} \oint (x dy - y dx) \dots \text{--- (i)}$$

The parametric form of a circle:

$$x = a \cos \theta, \quad y = a \sin \theta$$

$$dx = a(-\sin \theta) d\theta \quad dy = a \cos \theta d\theta$$

then (i) becomes,

$$A = \frac{1}{2} \oint [a \cos \theta \cdot a \cos \theta d\theta - a \sin \theta \cdot (-a \sin \theta) d\theta]$$

$$= \frac{1}{2} \oint a^2 \cos^2 \theta d\theta + a^2 \sin^2 \theta d\theta$$

$$= \frac{1}{2} \oint a^2 d\theta (\cos^2 \theta + \sin^2 \theta)$$

$$= \frac{a^2}{2} \oint d\theta$$

$$= \frac{a^2}{2} [\theta]_0^{2\pi}$$

$$= \frac{a^2}{2} \cdot 2\pi$$

$$= a^2 \pi$$

[∵ Note: Circle limit.
(0, 2π)]

J.M.

Q. Find the Area of astroid by using Green's theorem: $x^{2/3} + y^{2/3} = a^{2/3}$

⇒ By Green theorem

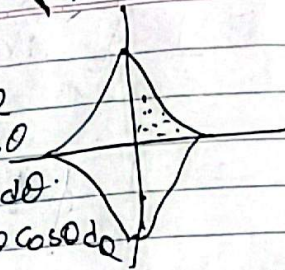
$$A = \oint (x dy - y dx)$$

$\sqrt{n+1} = n$ Date _____
Page _____

Parametric form:

$$x = a \cos^3 \theta; dx = -3a \cos^2 \theta \sin \theta d\theta$$

$$y = a \sin^3 \theta; dy = 3a \sin^2 \theta \cos \theta d\theta$$



Now

$$A = \int_0^{\pi/2} a \cos^3 \theta (3a \sin^2 \theta \cos \theta) d\theta + \int_{\pi/2}^{\pi} a \sin^3 \theta (-3a \cos^2 \theta \sin \theta) d\theta$$

$$= \int_0^{\pi/2} 3a^2 \cos^4 \theta \sin^2 \theta d\theta - \int_{\pi/2}^{\pi} 3a^2 \sin^4 \theta \cos^2 \theta d\theta$$

$$= \frac{1}{2} \int_0^{\pi} 3a^2 \cos^4 \theta \sin^2 \theta d\theta - \left[a \sin^3 \theta (-3a \cos^2 \theta \sin \theta) \right]_0^{\pi/2}$$

$$= \frac{1}{2} \int_0^{\pi} 3a^2 \cos^4 \theta \sin^2 \theta d\theta + \frac{1}{2} \int_0^{\pi} 3a^2 \sin^4 \theta \cos^2 \theta d\theta$$

$$= \frac{3a^2}{2} \int_0^{\pi} \cos^4 \theta + \frac{3a^2}{2} \int_0^{\pi} \sin^4 \theta$$

$$= \frac{1}{2} \int_0^{\pi} 3a^2 \cos^2 \theta \sin^2 \theta d\theta (\cos^2 \theta + \sin^2 \theta)$$

$$= \frac{3a^2}{2} \int_0^{\pi} \cos^2 \theta \sin^2 \theta d\theta$$

$$= \frac{3a^2}{2} \left[\frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5} \right]_0^{\pi}$$

$\sqrt{\frac{3}{2}} = \left(\frac{3}{2}-1\right) \sqrt{\frac{3}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ Date _____
Page _____

As, astroid, there are 4 Symmetrical Part. So taking first part only 0 to $\pi/2$

$$= 2 \times \frac{1}{2} \times 3a^2 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta$$

$$= 6a^2 \times \frac{\sqrt{\frac{2+1}{2}} \cdot \sqrt{\frac{2+1}{2}}}{2 \sqrt{\frac{2+2+2}{2}}} = \frac{6a^2 \left(\frac{3}{2}-1\right) \left(\frac{3}{2}-1\right) \left(\frac{3}{2}-1\right) \sqrt{\frac{3}{2}-1}}{2 \sqrt{3}}$$

$$= 6a^2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{3a^2 \sqrt{11}}{4} = \frac{3a^2 \sqrt{11}}{8}$$

Note: $\int_0^{\pi/2} \sin^n \theta \cos^m \theta d\theta = \frac{\sqrt{\frac{n+1}{2}} \cdot \sqrt{\frac{m+1}{2}}}{2 \sqrt{\frac{n+m+2}{2}}}$

① $\int_0^{\pi/2} \sin^n \theta d\theta = \frac{\sqrt{\frac{n+1}{2}}}{2 \sqrt{\frac{n+2}{2}}}$

Date _____
Page _____

$$\textcircled{3} \int_0^{\pi/2} \cos^m \theta d\theta = \frac{\left[\frac{m+1}{2} \right]}{2 \sqrt{\frac{m+2}{2}}}$$

if $n=0$

Evaluate the integral using Green's theorem.

$\textcircled{2}$ Use Green's theorem to evaluate the integral $\oint F \cdot d\vec{r}$.

$$\vec{F} = \text{grad}(\sin x \cos y) \quad C: \text{ellipse: } 25x^2 + 9y^2 = 225$$

$\Rightarrow$ Here,

$$\phi = \sin x \cos y$$

$$\text{grad } \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y}$$

$$= i \frac{\partial (\sin x \cos y)}{\partial x} + j \frac{\partial (\sin x \cos y)}{\partial y}$$

$$= \cos x \cos y \cdot i + (-\sin x \sin y) \cdot j$$

$$= \cos x \cos y \cdot i - \sin x \sin y \cdot j$$

Date _____
Page _____

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x \cos y & -\sin x \sin y & 0 \end{vmatrix}$$

$$= 0$$

So,

Note: Clockwise direction Ans: π change
 $+ve \pi$ -ve and -ve π $+ve$

Q. evaluate the integral using Green theorem.
 (1) $\oint_C [(2x-y+4)dx + (5y+3x-6)dy]$ around the
 circle $x^2 + y^2 = 16$ $[64\pi]$.

(2) $\int_C [(x^2 - 6\sin y)dx + (y + \sin x)dy]$ C: triangle
 $(0,0), (\pi,0), (\pi,1)$ $[11\cos 1 - \pi]$.

(3) Evaluate: $\oint_C (\vec{F} \cdot d\vec{r})$ Counter Clockwise $\vec{F} = \sin y, 6x$.
 C: $(0,0), (\pi,0), (\pi,1)$ $[11\cos 1 - \pi - 1]$

Q.2.1] Given,

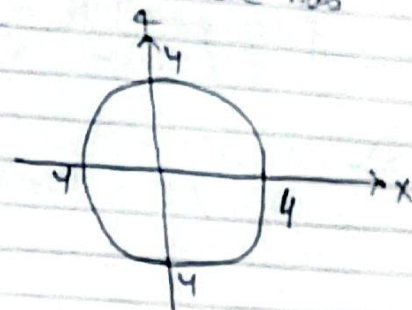
$$\oint_C (2x-y+4)dx + (5y+3x-6)dy$$

Comparing with $\int f_1 dx + f_2 dy$ then,

$$f_1 = 2x-y+4 \Rightarrow \frac{\partial f_1}{\partial y} = -1.$$

$$f_2 = 5y+3x-6 \Rightarrow \frac{\partial f_2}{\partial x} = 3$$

Surface: $x^2 + y^2 = 16$ which is circle has
 radius is 4.
 then.



Q.1] Evaluate the integral using Green's theorem
 $\oint_C (2x-y+4) dx + (5y+3x-6) dy$ around
 the circle $x^2+y^2=16$.

⇒ Given $I = \oint_C (2x-y+4) dx + (5y+3x-6) dy$ — (i)

Where C is the circle $x^2+y^2=16$

Comparing (i) with $\oint_C \vec{F} \cdot d\vec{r}$ then

$$\vec{F} = (2x-y+4)\vec{i} + (5y+3x-6)\vec{j}$$

then,

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y+4) & (5y+3x-6) & 0 \end{vmatrix}$$

$$= \vec{k} [3 - (-1)]$$

$$= 4\vec{k} \Rightarrow \text{curl } \vec{F} \cdot \vec{k} = 4\vec{k} \cdot \vec{k} = 4$$

As given path C is closed circle.

$x^2+y^2=16$ having radius 4.

and angular movement from 0 to 2π then
 By Green's theorem.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl } \vec{F} \cdot \vec{k}) dA$$

$$= \iint_R 4 dA \quad \text{--- (ii)}$$

Where $dA = dx dy$ then, from (ii).

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \int_0^4 4 r dr d\theta$$

$$= 4 \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^4 d\theta$$

$$= 4 \int_0^{2\pi} (8) d\theta$$

$$= 32 \pi$$

$$= 32\pi$$

Rough

$$\begin{aligned} & \frac{\pi}{2} \quad 4 \\ & 0 \quad 4 \quad 0 \\ & 4 \times 4 \left[\frac{\pi}{2} \right] \\ & 0 \quad 4 \quad 0 \\ & = 16 \left[\frac{\pi}{2} \right] \\ & = \frac{16\pi}{2} \times 4 \\ & = 32\pi \end{aligned}$$

2. $\int_C [(x^2 - \cosh y) dx + (y + \sin x) dy]$ C: triangle.
 $(0,0), (\pi,0), (\pi,1)$.

$\Rightarrow$ here. Given,
 $\int_C [(x^2 - \cosh y) dx + (y + \sin x) dy] \quad \text{--- (i)}$

Comparing (i) with the integral $\int_C f_1 dx + f_2 dy$.
 then.

$$f_1 = x^2 - \cosh y \Rightarrow \frac{\partial f_1}{\partial y} = -(\sinh y)$$

$$= \sinh y.$$

$$f_2 = y + \sin x \Rightarrow \frac{\partial f_2}{\partial x} = \cos x.$$

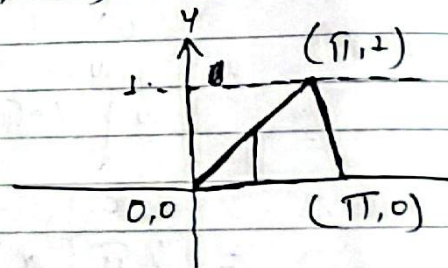
C: triangle $(0,0), (\pi,0), (\pi,1)$.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y = \frac{1}{\pi} \cdot x.$$

$$= x/\pi \quad \text{then,}$$

$$y = 0 \text{ to } y = x/\pi \text{ \& } x = 0 \text{ to } x = \pi$$



By Green's theorem,

$$\int_C f_1 dx + f_2 dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

then,

$$\int_C [(x^2 - \cosh y) dx + (y + \sin x) dy] = \iint_R (\cos x - \sinh y) dA$$

$$= \int_0^{\pi} \int_0^{x/\pi} (\cos x - \sinh y) dy dx$$

$$= \int_0^{\pi} \left[\cos x \left(\frac{y}{\pi} \right) - (-\cosh y) \right]_0^{x/\pi} dx$$

$$= \int_0^{\pi} \left[\cos x \cdot \left(\frac{x}{\pi} \right) + \cosh \frac{x}{\pi} - \cosh 0 \right] dx$$

1.10

Date: _____
Page: _____

Q. Evaluate the integral using Green's theorem
 $\oint_C (2x - y + 4) dx + (5y + 3x - 6) dy$ around
 the circle $x^2 + y^2 = 16$

⇒ Given, $\oint_C (2x - y + 4) dx + (5y + 3x - 6) dy$ — (i)

Comparing (i) with integral $\oint_C F_1 dx + F_2 dy$ then,

$$F_1 = 2x - y + 4 \Rightarrow \frac{\partial F_1}{\partial y} = -1$$

$$F_2 = 5y + 3x - 6 \Rightarrow \frac{\partial F_2}{\partial x} = 3$$

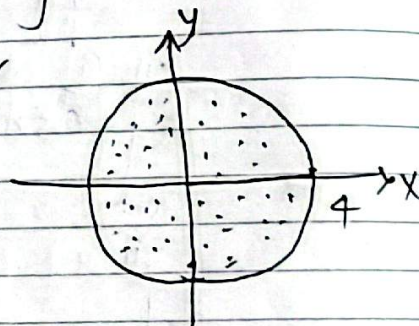
Surface is a circle: $x^2 + y^2 = 16$ which
 has radius 4. The
 area element in polar
 coordinate is

$$dA = r dr d\theta$$

So,

$$r = 0 \text{ to } r = 4$$

$$\theta = 0 \text{ to } \theta = 2\pi$$



By Green's theorem,

$$\oint_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$\Rightarrow \oint_C [(2x - y + 4) dx + (5y + 3x - 6) dy] = \iint_R [3 - (-1)] dA$$

$$= \iint_R 4 dA$$

$$2\pi \cdot 4$$

$$= \int_0^{2\pi} \int_0^4 4r dr d\theta$$

$$\text{Curl } \vec{F} = 0$$

$$= 4 \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^4 d\theta$$

$$= 4 \int_0^{2\pi} \left(\frac{16}{2} - 0 \right) d\theta$$

$$= 32 \left[\theta \right]_0^{2\pi}$$

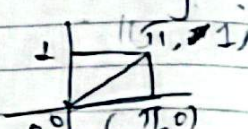
$$\vec{F} \cdot \nabla \phi$$

$$= 32 (2\pi - 0)$$

$$= 64\pi$$

||

2. $\oint_C [(x^2 - \cosh y) dx + (y + \sin x) dy]$ C: triangle
 $(0,0), (\pi,0), (\pi,1), (0,1)$



$\Rightarrow$ Given: $\oint_C (x^2 - \cosh y) dx + (y + \sin x) dy$ — (1)

Comparing (1) with $\int f_1 dx + f_2 dy$ then,

$$f_1 = x^2 - \cosh y \Rightarrow \frac{\partial f_1}{\partial y} = -\sinh y$$

$$f_2 = y + \sin x \Rightarrow \frac{\partial f_2}{\partial x} = \cos x$$

C: triangle is $(0,0), (\pi,0), (\pi,1)$ then.

$$x=0 \text{ to } x=\pi$$

For a fixed x , y range 0 to 1. then

now, $y=0$ to $y=1$.

Green's theorem:

$$\oint_C f_1 dx + f_2 dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA.$$

$$= \iint_R (\cos x + \sinh y) dx dy$$

$$= \int_0^1 \int_0^\pi (\cos x + \sinh y) dy dx.$$

$$= \int_0^1 \left[\int_0^\pi \cos x dy + \int_0^\pi \sinh y dy \right] dx.$$

$$= \int_0^1 [\cos x (1-0) + \cosh(1) - \cosh(0)] dx.$$

$$= \int_0^1 [\cos x + \cosh(1) - 1] dx.$$

$$= \int_0^1 \cos x dx + \int_0^1 [\cosh(1) - 1] dx.$$

$$= [\sin x]_0^1 + [\cosh(1) - 1] [x]_0^1.$$

$$= 0 + [\cosh(1) - 1] (\pi - 0)$$

$$= \pi [\cosh(1) - 1]$$

$$= \pi \cosh(1) - \pi$$

$\Rightarrow$

3. Evaluate $\oint \vec{F} \cdot d\vec{r}$ Counter Clockwise $\vec{F} = \sin y, \cos x$
 $C: (0,0), (\pi,0), (\pi,1)$

$\Rightarrow$ Given $\vec{F} = (\sin y, \cos x) = \sin y \vec{i} + \cos x \vec{j}$

$$\text{Curl } \vec{F} = \nabla \cdot \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & \cos x & 0 \end{vmatrix}$$

$$= (-\sin x - \cos y) \cdot \vec{k}$$

$$\text{Curl } \vec{F} \cdot \vec{k} = (-\sin x - \cos y) \cdot \vec{k} \cdot \vec{k}$$

$$= -\cos y - \sin x$$

Counter is $(0,0), (\pi,0), (\pi,1)$

$x = 0$ to $x = \pi$

for a fixed x, y ranges from 0 to $1 - \frac{x}{\pi}$

then, $y = 0$ to $y = \frac{\pi}{\pi}$

Volume integral:

Gauss divergence theorem:

Statement: Let T be the closed path and bounded region in space whose boundary is piecewise smooth orientable surface. Let $\vec{F} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$ be vector valued function then,

$$\iiint_S \vec{F} \cdot \vec{n} dA = \iiint_T \text{div } \vec{F} \cdot dV$$

$\nabla \cdot \vec{F}$ P.V.

Q. Find the integral $\iiint_S \vec{F} \cdot \vec{n} dA$; $\vec{F} = 4x^2 \vec{i} + 2y^2 \vec{j} + z^2 \vec{k}$ and S is the surface of cube $|x| \leq 1, |y| \leq 1, |z| \leq 1$.

$\Rightarrow$ Given.

$$\vec{F} = 4x^2 \vec{i} + 2y^2 \vec{j} + z^2 \vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (4x^2 \vec{i} + 2y^2 \vec{j} + z^2 \vec{k})$$

$$= (4 + 4y + 2z)$$

Since the surface is a cube

$$-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1.$$

Now. By Gauss divergence theorem.

$$\oint_S (\vec{F} \cdot \vec{n}) dA = \iiint_V \text{div } \vec{F} \cdot dV \quad \text{--- (1)}$$

Where $dV = dx dy dz$. then,
from (1),

$$\oint (\vec{F} \cdot \vec{n}) dA = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (4 + 4y + 2z) dx dy dz$$

$$= \int_{-1}^1 \int_{-1}^1 \left[4x + 4yx + 2zx \right]_{-1}^1 dy dz.$$

$$= \int_{-1}^1 \int_{-1}^1 [4 + 4y + 2z + 4 + 4y + 2z] dy dz$$

$$= \int_{-1}^1 \int_{-1}^1 (8 + 8y + 4z) dy dz$$

$$= \int_{-1}^1 \left[8y + \frac{8y^2}{2} + 4zy \right]_{-1}^1 dz$$

$$= \int_{-1}^1 \left[8 + 4 + 4z - (8 + \frac{8}{2} - 4z) \right] dz$$

$$= \int_{-1}^1 (8 + 4 + 4z + 8 - 4 - 4z) dz$$

$$= \int_{-1}^1 (12) dz$$

$$= 12 \left[z \right]_{-1}^1$$

$$= 12 \left[1 - (-1) \right]$$

$$= 12 \times 2$$

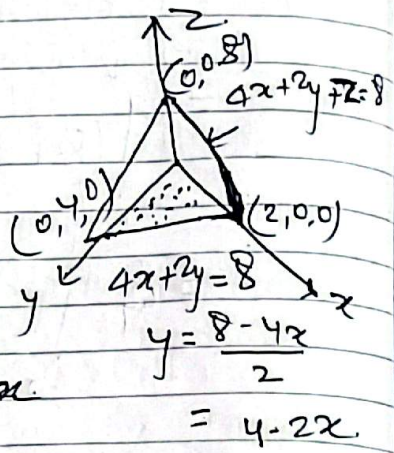
$$= 24$$

Ex. 6 (B)

Evaluate $\iiint_V \phi \, dV$, where $\phi = 45xy$ and V be the closed region enclosed by planes: $4x+2y+z=8$, $x=0$, $y=0$, $z=0$

⇒ According to question

$$\int_0^2 \int_0^{8-4x} \int_0^{8-4x-2y} 45xy \, dz \, dy \, dx$$



$$= \int_0^2 \int_0^{8-4x} 45xy \left[z \right]_0^{8-4x-2y} dy \, dx$$

$$= \int_0^2 \int_0^{4-2x} 45xy (8-4x-2y) dy \, dx$$

$$= \int_0^2 \int_0^{4-2x} (360x^2y - 180x^3y - 90xy^2) dy \, dx$$

$$= \int_0^2 \left[360x^2 \left(\frac{y^2}{2} \right) - 180x^3 \left(\frac{y^2}{2} \right) - 90x \left(\frac{y^3}{3} \right) \right]_0^{4-2x} dx$$

$$= \int_0^2 \left[180x^2(4-2x)^2 - 90x^3(4-2x)^2 - 30x^2(4-2x)^3 \right] dx$$

$$= \int_0^2 \dots dx$$

Stokes' theorem :-

Statement: Let S be a piecewise smooth oriented surface in space and the boundary of S be a piecewise smooth simple closed curve C . Let $\vec{F}(x, y, z)$ be continuous vector valued funcⁿ. then.

$$\iint_S (\text{curl } \vec{F} \cdot \vec{n}) \, dS = \oint_C \vec{F} \cdot d\vec{r}$$

$$\text{Where } \vec{n} = \vec{N} = \vec{r}_x \times \vec{r}_y = \vec{r}_y \times \vec{r}_x$$

Ex. Evaluate the integral $\oint_C \vec{F} \cdot d\vec{r}$ by using Stoke's theorem where $\vec{F}(y^2, z^2, x^2)$ and $C: x+y+z=1$ in $z \geq 0, y \geq 0, x \geq 0$.

$$\Rightarrow \text{Given, } \vec{F} = (y^2, z^2, x^2)$$

We know that: by Stoke's theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F} \cdot \vec{n}) \, dS$$

$$\text{Now, } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= x\vec{i} + y\vec{j} + (1-x-y)\vec{k} \quad [\because x+y+z=1]$$

for $\vec{n}$,

$$\vec{r}_x = \vec{i} - \vec{k}$$

$$\vec{r}_y = \vec{j} - \vec{k}$$

$$\vec{n} = \vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \vec{i} + \vec{j} + \vec{k}$$

$$\vec{F} = y^2\vec{i} + z^2\vec{j} + x^2\vec{k}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix}$$

$$= \vec{i}(-2z) + \vec{j}(2x) + \vec{k}(-2y)$$

$$= -2z\vec{i} + 2x\vec{j} - 2y\vec{k}$$

$$\text{and } \text{curl } \vec{F} \cdot \vec{n} = (-2z\vec{i} + 2x\vec{j} - 2y\vec{k}) \cdot (\vec{i} + \vec{j} + \vec{k})$$

$$= -2z + 2x - 2y$$

$$\oint_C \vec{F} \cdot d\vec{r} = -2(x+y+z)$$

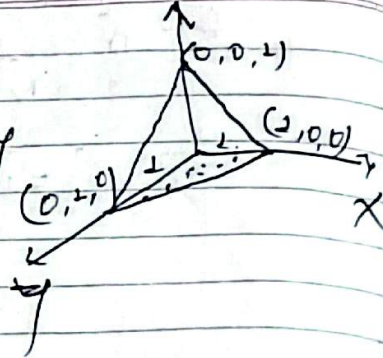
$$= -2$$

From figure,

$$\iint_S (\text{curl } \vec{F} \cdot \vec{n}) dS = \iint -2 dx dy$$

$$= -2 \times \frac{1}{2} \times 1 \times 1$$

$$= -1$$



v.v.v. Evaluate the integral by using Stoke's theorem $\oint_C (4y dx + 2z dy + by dz)$ C:

Curve intersection of $x^2 + y^2 + z^2 = 6z$
 & $z = x + 3$.

$\Rightarrow$ we know that, By Stoke's theorem.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F} \cdot \vec{n}) dS$$

Now,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= x\vec{i} + y\vec{j} + (x+3)\vec{k}$$

For $\vec{n}$,

$$\vec{r}_x = -\vec{i} + \vec{k}, \quad \vec{r}_y = \vec{j}$$

Now,

$$\vec{n} = \vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -\vec{i} + \vec{k}$$

Let $\vec{F} = 4y dx + 2z dy + by dz$

$$= 4y\vec{i} + 2z\vec{j} + by\vec{k}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4y & 2z & by \end{vmatrix}$$

$$= \vec{i}(b-2) + \vec{j}(0) + \vec{k}(0-4)$$

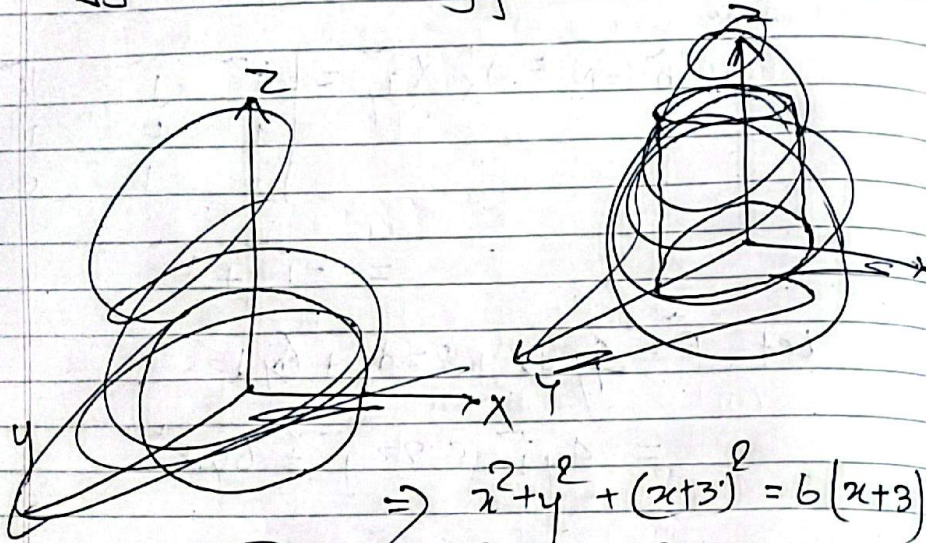
$$= 4\vec{i} - 4\vec{k}$$

$$\text{curl } \vec{F} \cdot \vec{n} = (4\vec{i} - 4\vec{k}) \cdot (-\vec{i} + \vec{k})$$

$$= -4 - 4 = -8$$

Now,

$$\iint_S (\text{curl } \vec{F} \cdot \vec{n}) \, dS = \iint -8 \, dx \, dy \quad \text{--- (1)}$$



$$\Rightarrow x^2 + y^2 + (x+3)^2 = 6(x+3)$$

Now, from (1), $\Rightarrow x^2 + y^2 + x^2 + 6x + 9 = 6x + 18$

$$= \iint -8 \, dx \, dy \Rightarrow 2x^2 + y^2 = 9$$

$$= -8 \iint \frac{x^2}{(\frac{9}{2})} + \frac{y^2}{(\frac{9}{1})} = 1$$

$$= -8 \times \pi \frac{3}{\sqrt{2}} \cdot 3 \Rightarrow \text{Area} = \pi ab.$$

$$= -9 \cdot 2 \pi \frac{3}{\sqrt{2}} \quad a = \frac{3}{2}, \quad b = \frac{3}{1}$$

$$= -9 \pi \cdot 4 \cdot \sqrt{2} \cdot \sqrt{2} / \sqrt{2} = -36 \pi \sqrt{2} \quad \#$$

Area not in negative then.

$$= 36 \sqrt{2} \pi \quad \#$$

Q. Find $\vec{F} \cdot d\vec{r}$ by using Stokes's theorem
where $\vec{F} = (2y^2, x, -z^3)$ and $C: x^2 + y^2 = a^2$
 $z = b \ (b > 0).$

$\Rightarrow$ We know that,

By Stokes's theorem.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F} \cdot \vec{n}) \, dS.$$

Now

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= x\vec{i} + y\vec{j} + b\vec{k}$$

$$\vec{r}_x = \vec{i}, \quad \vec{r}_y = \vec{j}$$

Now,

$$\vec{n} = \vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x/2 & 2xy & 2y^2 \\ y^2 & x & -x^3 \end{vmatrix}$$

$$= 0 + \vec{j}(0) + \vec{k}(1-4y)$$

$$= (1-4y)\vec{k}$$

Now,

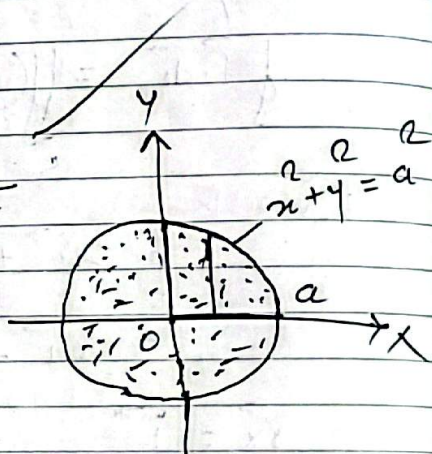
$$\text{Curl } \vec{F} \cdot \vec{n} = (1-4y)\vec{k} \cdot \vec{k}$$

$$= 1-4y$$

$$x=0 \text{ to } x=a$$

$$y=0 \text{ to } y=\sqrt{a^2-x^2}$$

Now Taking one part.



$$\int \vec{F} \cdot d\vec{r} = \iint (1-4y) dy dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[y - 4y^2 \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\sqrt{a^2-x^2} - 2(a^2-x^2) \right] dx$$

$$= \int_0^a \left(\sqrt{a^2-x^2} - 2a^2 + 2x^2 \right) dx$$

By Polar Condition,

$$x = r \cos \theta, y = r \sin \theta$$

$$r = 0 \text{ to } a$$

$$\theta = 0 \text{ to } 2\pi$$

Now,

$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \int_0^a (1-4r \sin \theta) r dr d\theta$$

$$= \int_0^{2\pi} \left[\theta - 4r^2 \sin \theta \right]_0^a d\theta$$

$$= \int_0^{2\pi} (2\pi - 4r \sin \theta) d\theta \Rightarrow 2\pi - 4r \sin \theta$$

$$\begin{aligned}
 &= \int_0^a \left(2\pi + 4r \cos \theta \right) r dr \\
 &= \int_0^a \left(2\pi r + 4r^2 \cos \theta \right) dr \\
 &=
 \end{aligned}$$

1. The necessary and sufficient for the vector valued function $\vec{a}$ of the scalar variable t to have a constant magnitude is $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$

2. Let $\vec{a} = 12 \cos 2t \vec{i} - 8 \sin 2t \vec{j} + 16t \vec{k}$ be the acceleration of the particle at any time t , find the velocity $\vec{v}$ and displacement $\vec{r}$ at time t and $\vec{r} = 0 = \vec{v}$ at $t=0$.

3. Show that it is exact and evaluate:

$$\int_{(0,2,3)}^{(2,2,2)} (yz \sinh(xz) dx + \cosh(xz) dy + xyz \sinh(xz) dz)$$

1. # Here.

Necessary condition:

Let $\vec{a}$ has a constant magnitude. Then, we have to show $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$.

We know,

$$\vec{a} \cdot \vec{a} = (|\vec{a}|)^2$$

Differentiating w. r. to t .

$$\frac{d}{dt}(\vec{a} \cdot \vec{a}) = \frac{d}{dt}(|\vec{a}|)^2$$

$$\Rightarrow \vec{a} \cdot \frac{d\vec{a}}{dt} + \vec{a} \cdot \frac{d\vec{a}}{dt} = \frac{d}{dt}(|\vec{a}|)^2$$

$$\Rightarrow 2\vec{a} \cdot \frac{d\vec{a}}{dt} = 0 \quad \left[\because |\vec{a}| \text{ is constant} \right]$$

$$\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$$

Sufficient condition,

Let $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$ then we have to show, that $\vec{a}$ has a constant magnitude.

We know, $\vec{a} \cdot \vec{a} = (|\vec{a}|)^2$

Differentiating w. r. to t ,

$$\frac{d}{dt}(\vec{a} \cdot \vec{a}) = \frac{d}{dt}(|\vec{a}|)^2$$

$$2\left(\vec{a} \cdot \frac{d\vec{a}}{dt}\right) = 2 \cdot |\vec{a}| \frac{d}{dt}(|\vec{a}|)$$

$$0 = |\vec{a}| \frac{d}{dt}(|\vec{a}|) \quad \left[\because \vec{a} \cdot \frac{d\vec{a}}{dt} = 0 \right]$$

Since $\vec{a} \neq 0$ so, $|\vec{a}| \neq 0$ then,

$$|\vec{a}| \frac{d}{dt}(|\vec{a}|) = 0$$

$$\Rightarrow \frac{d}{dt}(|\vec{a}|) = 0$$

hence proved.

2. # Given,

$$\frac{d^2\vec{r}}{dt^2} = \vec{a} = 12\cos 2t \vec{i} - 8\sin 2t \vec{j} + 16t \vec{k} \quad \text{--- (1)}$$

be the acceleration at any t .

Also, given that $\vec{r} = 0 = \vec{v}$ at $t = 0$

$\vec{v}$ be the velocity at any time t , then

$$\vec{r} = 0 = \frac{d\vec{r}}{dt} \text{ at } t = 0.$$

Integrating (i) w. r. to t then,

$$\frac{d\vec{r}}{dt} = 6\sin 2t \vec{i} + 4\cos 2t \vec{j} + 8t^2 \vec{k} + C. \quad \text{--- (ii)}$$

at $t = 0$ then eqn (ii).

$$0 = 0 + 4\cos 2t \vec{j} + C.$$

$$0 = 4\vec{j} + C.$$

$$\Rightarrow C = -4\vec{j}$$

From eqn (ii), becomes.

$$\frac{d\vec{r}}{dt} = 6\sin 2t \vec{i} + 4\cos 2t \vec{j} + 8t^2 \vec{k} - 4\vec{j}$$

Again integrating,

$$\vec{r} = -3\cos 2t \vec{i} + 2\sin 2t \vec{j} + \frac{8t^3}{3} \vec{k} - 4t \vec{j} + C$$

At $t = 0$;

$$0 = -3\vec{i} + C$$

$$C = 3\vec{i}$$

then.

$$\vec{r} = -3\cos 2t \vec{i} + 2\sin 2t \vec{j} + \frac{8t^3}{3} \vec{k} - 4t \vec{j} + 3\vec{i}$$

$$\vec{r} = (3 - 3\cos 2t) \vec{i} + (-4t + 2\sin 2t) \vec{j} + \frac{8t^3}{3} \vec{k}$$

Hence,

$$\text{displacement } (\vec{r}) = (3 - 3\cos 2t) \vec{i} - (4t - 2\sin 2t) \vec{j} + \frac{8t^3}{3} \vec{k}$$

$$\text{velocity } (\vec{v}) = 6\sin 2t \vec{i} + 4\cos 2t \vec{j} + 8t^2 \vec{k} - 4\vec{j} \\ = 6\sin 2t \vec{i} - (4 - 4\cos 2t) \vec{j} + 8t^2 \vec{k}$$

Q. 23] $\Rightarrow$ Given integral is,

$$\int_{(0,2,3)}^{(1,1,1)} yz \sinh(xz) dz + \cosh(xz) dy + xy \sinh(xz) dx$$

Comparing the value of the integral $\textcircled{1}$ with $\int F_1 dx + F_2 dy + F_3 dz$ then,

$$F_1 = yz \sinh xz, \quad F_2 = \cosh(xz)$$

$$F_3 = xy \sinh(xz)$$

$$\text{then, } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \Rightarrow z \sinh xz = z \sinh(xz)$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \Rightarrow yz \cosh(xz) = y \cosh(xz)$$

$$xyz \cosh(xz) + y \sinh(xz) = xyz \cosh(xz) + y \sinh(xz)$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \Rightarrow x \sinh(xz) = x \sinh(xz)$$

it is exact.

$$\int = \int_{(0,2,3)}^{(1,1,1)} F_1 dx + \int_{(0,2,3)}^{(1,1,1)} (\text{term of } F_2 \text{ free from } x) dy + \int_{(0,2,3)}^{(1,1,1)} (\text{term of } F_3 \text{ free from } x, y) dz$$

$$= \int_{(0,2,3)}^{(1,1,1)} yz \sinh(xz) dx + \int_{(0,2,3)}^{(1,1,1)} 0 dy + \int_{(0,2,3)}^{(1,1,1)} 0 dz$$

$$= \int_{(0,2,3)}^{(1,1,1)} yz \sinh(xz) dx$$

$$= \left[yz \frac{\cosh xz}{z} \right]_{0,2,3}^{1,1,1}$$

$$= \left[\cancel{yz} \cdot \cosh(1) - \cancel{yz} \cosh(0.3) \right]$$

$$= \cosh(1) - 2$$

Q.4] If $\phi = x^3 + y^3 + z^3 - 3xyz$ find

(a) $\text{div}(\text{grad } \phi)$ (b) $\text{Curl}(\text{grad } \phi)$.

$\Rightarrow$ Given, $\phi = x^3 + y^3 + z^3 - 3xyz$.

(c) $\text{div}(\text{grad } \phi) = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (\text{grad } \phi)$

— (i)

$$\text{grad } \phi = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\vec{i} + (3y^2 - 3xz)\vec{j} + (3z^2 - 3xy)\vec{k}$$

Now,

$$\text{div}(\text{grad } \phi) = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot [(3x^2 - 3yz)\vec{i} + (3y^2 - 3xz)\vec{j} + (3z^2 - 3xy)\vec{k}]$$

$$= 6x + 6y + 6z$$

#

(d) $\text{Curl}(\text{grad } \phi) = \nabla \times (\text{grad } \phi)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= \vec{i} [(-3z) - (-3z)] + \vec{j} [(-3x) - (-3x)] + \vec{k} [(-3y) - (-3y)]$$

$$= \vec{i} [-3z - (-3z)] + \vec{j} [-3x - (-3x)] + \vec{k} [-3y - (-3y)]$$

$$= \vec{0}$$

Q.7.5] Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ if $\vec{F} = 4xy\vec{i} + 8y\vec{j} + 2z\vec{k}$ along the curve $y=2x$, $z=2x$ from $(0,0,0)$ to $(3,6,6)$.

$\Rightarrow$ Given, $\vec{F} = 4xy\vec{i} + 8y\vec{j} + 2z\vec{k}$.

Curve, $y=2x$, $z=2x$.

Since, we have $\vec{r} = (x, y, z) = (x, 2x, 2x)$ so,

$$d\vec{r} = (1, 2, 2) dx$$

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= (4x, 8y, 2) \cdot (1, 2, 2) dx \\ &= (8x^2, 16x, 2) \cdot (1, 2, 2) dx \\ &= (8x^2 + 32x + 4) dx\end{aligned}$$

Now, integral of $\vec{F}$ along $y=2x, z=2x$ from $(0,0,0)$ to $(3,6,6)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^3 (8x^2 + 32x + 4) dx$$

$$= \left[\frac{8x^3}{3} + \frac{32x^2}{2} + 4x \right]_0^3$$

$$= \left[\frac{8}{3} \cdot 3^3 + \frac{32}{2} \cdot 3^2 + 12 \right]$$

$$= 72 + 144 + 12$$

$$= 228$$

(b) Evaluate: $\iint_S \vec{F} \cdot \vec{n} dA$, where $\vec{F} = 12xy\vec{i} - 3yz\vec{j} + 2z\vec{k}$, S is the portion of plane $x+y+z=1$ for $x \geq 0, y \geq 0, z \geq 0$

Here, Given $\vec{F} = 12xy\vec{i} + 3yz\vec{j} + 2z\vec{k}$.
Surface: $x+y+z=1$ in $x, y, z \geq 0$

$$\text{Let } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= x\vec{i} + y\vec{j} + (1-x-y)\vec{k}$$

$$\vec{r}_x = \vec{i} - \vec{k}, \quad \vec{r}_y = \vec{j} - \vec{k}$$

$$\vec{n} = \vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \vec{i} + \vec{j} + \vec{k}$$

$$\vec{F} \cdot \vec{n} =$$

7. Fourier series (8+7=15 marks)

Define periodic funcⁿ:

⇒ A funcⁿ $f(x)$ is said to be periodic if $f(x+p) = f(x)$ where p is the smallest period of $f(x)$.

Ex. 1] Find the smallest period of $\sin x$

$f(x) = \sin x$ Now by defⁿ:

$$f(x) = f(x+p)$$

$$\Rightarrow \sin x = \sin(x+p)$$

$$\Rightarrow \sin x = \sin x \cos p + \cos x \sin p$$

Equating the coefficient,

$$1 = \cos p$$

$$\cos 2\pi = \cos p$$

$$\therefore p = 2\pi$$

Q. Find the sm.

$$\tan(kx)$$

Let $f(x) = \tan kx$

We know that,

$$f(x) = f(x+p)$$

$$\tan kx = \tan k(x+p)$$

$$\tan kx = \tan(kx + kp)$$

$$= \tan kx + \tan kp$$

$$\tan kx = \tan kx + \tan kp$$

$$\tan kx = \tan kx + \tan kp$$

$$\tan kx = \tan kx + \frac{\tan kp}{1 - \tan kx \tan kp}$$

$$\text{Equating, } 1 = \frac{\tan kp}{1 - \tan kx \tan kp}$$

$$0 = \frac{\tan kp}{1 - \tan kx \tan kp}$$

From (i) and (ii),

$$0 = \tan k p$$

$$\tan \pi = \tan k p$$

$$k p = \pi$$

$$p = \pi/k$$

Odd & even funcⁿ:

The funcⁿ $f(x)$ said to be odd funcⁿ if $f(-x) = -f(x)$ eg. $\sin x, \tan x, x, x^3, x^5$

even:

The funcⁿ $f(x)$ said to be even funcⁿ if $f(-x) = f(x)$ eg. $\cos x, \sec x, x^2, x^4$ etc

Properties of definite integral:

(i) $\int_a^a f(x) dx = 0$ if $f(x)$ is odd.

$$\int_{-\pi}^{\pi} x \cos x dx = 0$$

(ii)

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(x) \text{ is even.}$$

$$\text{Ex. } \int_{-1}^1 x dx = \left[\frac{x^2}{2} \right]_{-1}^1 = \frac{2}{2} = 1$$

$$= 2 \left[\frac{x^2}{2} \right]_0^1 = 2 \cdot \frac{1}{2} = 1$$

$$\int_{-a}^a f(-x) dx = - \int_a^0 f(x) dx + \int_0^a f(x) dx = \int_a^0 f(x) dx + \int_0^a f(x) dx = \int_{-a}^a f(x) dx$$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

0

* find Fourier series of $f(x) = x$ $- \pi < x < \pi$

⇒ we know that,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

— (i)

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx = \frac{1}{2\pi} \left[\frac{x^2}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{(\pi)^2}{2} - 0 \right]$$

$$= \frac{\pi^2}{4\pi} = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \cos nx dx$$

$$= \frac{1}{\pi} \left[\frac{x \sin nx}{n} - \frac{x(-\cos nx)}{n^2} + \frac{2(-\sin nx)}{n^3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{x \sin nx}{n} + \frac{x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[0 + \frac{2\pi \cdot 1}{n^2} - 0 - (0 + 0 - 0) \right]$$

$$= \frac{1}{\pi} \cdot \frac{2\pi}{n^2}$$

$$= \frac{2}{n^2}$$

$$\text{Again, } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx$$

$$= \frac{1}{\pi} \left[\frac{x(-\cos nx)}{n} - \frac{x(-\sin nx)}{n^2} + \frac{2 \cos nx}{n^3} \right]_{-\pi}^{\pi}$$

"

Fourier Cosine and Sine Series [8-marks]

we have, ∞
 $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ — (i)

$(-\pi \text{ to } \pi)$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Let $f(x)$ is even funcⁿ [Cosine] -

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0$$

Hence Cosine funcⁿ:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

Let $f(x)$ is odd funcⁿ [Sine]

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

Hence, $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Find the sine & cosine funcⁿ of $f(x) = 1$
for $0 < x < l$

⇒ Since.

$$f(x) = 1, \text{ for } 0 < x < l$$

for Fourier sine funcⁿ:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right) \quad \text{--- (i)}$$

$$b_n = \frac{2}{l} \int_0^l 1 \cdot \sin\left(\frac{n\pi}{l}x\right) dx$$

$$= \frac{2}{l} \left[\frac{-\cos\left(\frac{n\pi}{l}x\right) \cdot x}{\left(\frac{n\pi}{l}\right)} \right]_0^l$$

$$= \frac{2}{l} \left[-\frac{\cos n\pi}{\frac{n\pi}{l}} + \frac{1}{\frac{n\pi}{l}} \right]$$

$$= \frac{2}{n\pi} [1 - \cos n\pi]$$

Required solⁿ is,

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - \cos n\pi] \cdot \sin\left(\frac{n\pi}{l}x\right) \quad \times$$

for Fourier series cosine funcⁿ:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) \quad \text{--- (ii)}$$

$$a_0 = \frac{1}{l} \int_0^l f(x) dx$$

$$= \frac{1}{l} \left[x \right]_0^l$$

$$= \frac{1}{l} [l - 0]$$

$$= 1$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \cos\left(\frac{n\pi}{l}x\right) dx$$

$$= \frac{2}{l} \left[\frac{\sin\left(\frac{n\pi}{l}x\right) \cdot x}{\left(\frac{n\pi}{l}\right)} \right]_0^l$$

$$= \frac{2}{l} \left[\frac{\sin n\pi}{\frac{n\pi}{l}} - 0 \right] = \frac{2 \sin n\pi}{n\pi}$$

$$= 0 \quad \because \text{since } \sin n\pi = 0$$

require.

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{2 \sin n\pi}{n\pi} \cdot \cos \frac{n\pi}{2} \cdot x$$

#

$$= 1 \quad \#$$

(i) $f(x) = \pi - x$ for $0 < x < \pi$.

(ii) $f(x) = e^x$ for $0 < x < 1$.

(iii) $f(x) = x^2$ for $0 < x < 2$.

Unit-2 } Power Series Solution [8+7=15]

Power series:

⇒ The infinite series of $f(x)$ is said to be power series if

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{ where } a_n \text{ is}$$

The coefficient of the series x_0 is the center of the series.

If $x_0 = 0$ then,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{--- (i)}$$

Q. Solve the differential eqn: $y' - 9y = 0$

⇒ let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$

eqn (i) be the solution of given diff. eqn.

Diff. y w.r. to x we get,

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

Then the original eqⁿ becomes

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots = 9a_0 + 9a_1x + 9a_2x^2 + 9a_3x^3 + \dots$$

Equating,

$$a_1 = 9a_0$$

$$2a_2 = 9a_1$$

$$a_2 = \frac{9}{2} \cdot 9a_0 = \frac{81a_0}{2}$$

$$3a_3 = 9a_2$$

$$a_3 = \frac{9}{3} \cdot \frac{81a_0}{2}$$

$$= \frac{729}{6} a_0$$

Q. $y'' + 9y = 0$

⇒ The general soln:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

Diff. eqⁿ ① w.r.t. x → ②

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \dots$$

Now, •

$$y'' + 9y = 0$$

$$y'' = -9y$$

$$2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \dots$$

$$= -9a_0 - 9a_1x - 9a_2x^2 - 9a_3x^3 - 9a_4x^4 - 9a_5x^5 - \dots$$

Equating, $x^0, x^1, x^2, x^3, x^4, x^5, \dots$

$$2a_2 = -9a_0, \quad 6a_3 = -9a_1$$

$$a_2 = -\frac{9}{2}a_0, \quad a_3 = -\frac{9}{6}a_1$$

$$12a_4 = -9a_2$$

$$a_4 = \frac{-9}{12}a_2$$

$$= \frac{-9}{12} \left(-\frac{9}{2}a_0 \right)$$

$$= \frac{81}{24}a_0$$

State Legendre's Eqⁿ and establish the solⁿ of Legendre's diff. eqⁿ. [8-marks].

⇒ The diff. eqⁿ of the form $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$.

Where n is real number is called L. diff. eqⁿ.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$
be the solution of Legendre's diff. eqⁿ.

Dirichlet's Integrals:

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1.$$

$$\iiint_V (x^{\alpha-1} \cdot y^{\beta-1} \cdot z^{\gamma-1}) dx dy dz.$$

Let V be a closed region in first octant in $\mathbb{R}^3$, and is bounded by surface $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$ and coordinate planes.

then the integral of the form

$$\iiint_V (x^{\alpha-1} \cdot y^{\beta-1} \cdot z^{\gamma-1}) dx dy dz \text{ is called}$$

Dirichlet's integral. where all a, b, c, p, q, r are positive.

$$\iiint_V (x^{\alpha-1} \cdot y^{\beta-1} \cdot z^{\gamma-1}) dx dy dz = \frac{\Gamma(\alpha) \cdot \Gamma(\beta) \cdot \Gamma(\gamma)}{\Gamma(\alpha + \beta + \gamma + 1)}$$

proved.

Example-8:

Put $\frac{x}{a} = u, \frac{y}{b} = v, \frac{z}{c} = w$
 $dx = a du, dy = b dv, dz = c dw$

then the tetrahedron $OABC$ is bounded by $u \geq 0, v \geq 0, w \geq 0$ and $u + v + w \leq 1$.
 So, volume of tetrahedron is

$$\begin{aligned} V &= \iiint_V dx dy dz = \iiint_V abc du dv dw \\ &= abc \iiint_V du dv dw \\ &= abc \iiint_V (u^{1-1} \cdot v^{1-1} \cdot w^{1-1}) du dv dw \end{aligned}$$

using Dirichlet's theorem,

$$\begin{aligned} V &= abc \frac{\Gamma(1) \cdot \Gamma(1) \cdot \Gamma(1)}{\Gamma(1+1+1+1)} = abc \frac{1 \times 1 \times 1}{\Gamma(4)} \\ &= \frac{abc}{3!} = \frac{abc}{6} \end{aligned}$$

$$dx = \frac{a}{p} \frac{dy}{y^2}$$

$$dx = \frac{a}{p} u^{\frac{1}{p}-1} du, \quad dy = \frac{b}{q} v^{\frac{1}{q}-1} dv$$

$$dz = \frac{c}{\gamma} \cdot \omega^{1/\gamma-1} \cdot d\omega$$

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \leq 1 \text{ with all +ve.}$$

Now,

$$V = \iiint_V (x^{l-1} \cdot y^{m-1} \cdot z^{n-1}) \, dx \, dy \, dz.$$

$$= \iiint_V (a u^{\frac{1}{p-1}})^{p-1} \cdot (b v^{\frac{1}{q-1}})^{q-1} \cdot (c w^{\frac{1}{r-1}})^{r-1} \cdot \frac{a}{p} u^{\frac{1}{p-1}} dv \cdot \frac{b}{q} v^{\frac{1}{q-1}} dv \cdot \frac{c}{r} w^{\frac{1}{r-1}} dw$$

$$= \frac{a^p b^m c^n}{pqr} \quad \begin{array}{c} \downarrow \downarrow \downarrow \\ b \end{array} \quad u^{p/p-1} \cdot v^{m/q-1} w^{n/r-1} du dv dw$$

$$V = \frac{a^p b^m c^n}{pqr} \cdot \frac{\sqrt{p/p} \cdot \sqrt{m/q} \cdot \sqrt{n/r}}{\sqrt{p + m/q + n/r + 2}}$$

Ex. 10

Given. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Prd.

$$\frac{x}{a} = u, \frac{y}{b} = v, \frac{z}{c} = w \text{ then}$$

$$dx = a du, \quad dy = b dv, \quad dz = c dw$$

And the given region is bounded by
 $u \geq 0$, $v \geq 0$, $w \geq 0$ and $u+v+w \leq 1$.

Now.

$$\text{volume } (V) = \int \int \int_V xyz \, dx \, dy \, dz.$$

$$= \int \int \int a^2 b^2 c^2 \, da \, db \, dc$$

$$= \int \int \int a^3 b^2 c^2 y^2 v w \, du \, dv \, dw$$

$$= \int_0^3 \int_0^2 \int_0^1 u^{3-1} \cdot v^{2-1} \cdot w^{2-1} du dv dw.$$

By Dirichlet's theorem. We get,

$$V = a^3 b^2 c^2 \frac{\sqrt{3 \cdot 2 \cdot 2}}{\sqrt{(3+2+2+1)}} \\ = a^3 b^2 c^2 \cdot \frac{2 \times 1 \times 1}{7!} = \frac{2a^3 b^2 c^2}{7!}$$

$$\frac{1}{2} \int_0^3 (2 + 6z) dz \\ = \frac{1}{2} \left[2z + 6 \frac{z^2}{2} \right]_0^3 \\ = \frac{1}{2} [2 \cdot 3 + 3 \cdot 3^2] \\ = \frac{33}{2}$$

Example. Exercise:

Given

$$2. \iiint_V (xy + yz + zx) dx dy dz \quad \text{--- (i)}$$

Here $x=0$ to $x=1$.

$y=0$ to $y=2$

$z=0$ to $z=3$ then from (i)

becomes,

$$V = \int_0^3 \int_0^2 \int_0^1 (xy + yz + zx) dx dy dz \\ = \int_0^3 \int_0^2 \left[4 \cdot \frac{x^2}{2} + yz \cdot x + z \frac{x^2}{2} \right]_0^1 dy dz \\ = \int_0^3 \int_0^2 \left[4 \cdot \frac{1}{2} + yz \cdot 1 + \frac{z}{2} - 0 \right] dy dz \\ = \frac{1}{2} \int_0^3 \int_0^2 (y + 2yz + z) dy dz \\ = \frac{1}{2} \int_0^3 \left[\frac{y^2}{2} + \frac{2y^2 z}{2} + zy \right]_0^2 dz \\ = \frac{1}{2} \int_0^3 (2 + 4z + 2z) dz$$

Unit-8

Define first order differential Equation:

Let us consider the linear first order constant coefficient partial differential equation

$$ax + by + c = f(x, y) \quad \text{--- (i)}$$

for a, b and c are constant with $a^2 + b^2 > 0$.

Example: 1.

Given

$$4x + 4y - 4 = 0 \quad \text{--- (i)}$$

We first identify $a=1, b=1$ and $c=4$

The relation between the differential is

$$\frac{dx}{1} \pm \frac{dy}{1} = \frac{du}{y} \quad \text{--- (ii)}$$

From (ii) Taking first and second term.

$$\frac{dx}{1} = \frac{dy}{1}$$

integrating $x = y + A \quad \text{--- (x)}$

$$\therefore A = x - y$$

Again. Taking second and last term.

$$\frac{dy}{1} = \frac{dy}{y}$$

integrating,

$$y = \ln(y) + B$$

$$u = e^{y/B} = e^y C_1 \quad \because C_1 = e^{y/B}$$

$C_1 = \text{constant}$.

Since $u = u(x, y)$ so, let.

$$C_1 = f(A) = f(x - y)$$

Now, The general solution of (ii) is

$$u = u(x, y) = f(x - y) e^y$$

#

Ex-2:

Given equation is $4x + 4y - 4 = 0$ — (i)

i.e. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - 4 = 0$

Comparing with $a u_x + b u_y - c = 0$ then,

The characteristics eqⁿ is.

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{4}$$

It's solution is

$$x - y = c_1 \text{ and } x = \ln(u) + c_2$$

i.e. $u(x, y) = A e^y$

As given $u(x, 0) = 1$ then, $A = 1$.

$\therefore u(x, y) = e^y$ #

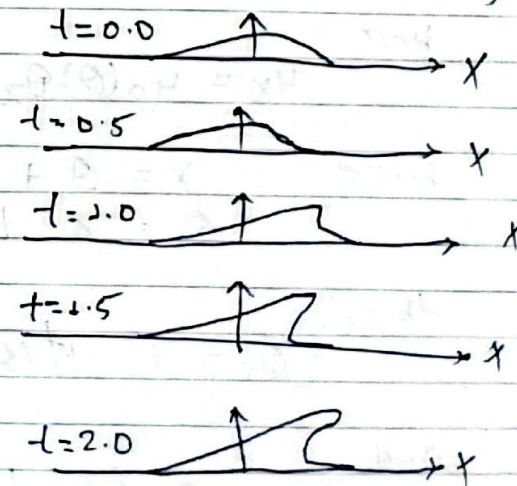
Application of PDE

1. Conservation laws [One Dimensional Conservation law]
2. Nonlinear Advection Equation.

$$u_t + c(u) \cdot u_x = 0$$

~~The~~ Breaking time: S.M.P.

The time at which the funcⁿ becomes multivalued is called the breaking time:



Consider,

$$u_t + c(u) u_x = 0$$
 — (i)

Set $x_0 = 0$ then the wave speed is.

$$f(\theta) = c(u_0(\theta))$$

The characteristics line is,

$$x = \theta + t f(\theta).$$

The solution of Equation (i) is.

$$u(x, t) = u(\theta + t f(\theta), t) \quad \text{--- (ii)}$$

$$= u_0(\theta) \text{ along } x = \theta + t f(\theta).$$

Here,

$$u_x = u'_0(\theta) \theta_x \text{ and } u_t = u'_0(\theta) \theta_t$$

Since,

$$x = \theta + t f(\theta)$$

$$\theta = x - t f(\theta)$$

then,

$$\theta_x = 1 - t f'(\theta) \theta_x \Rightarrow \theta_x = \frac{1}{1 + t f'(\theta)}$$

and

$$\theta_t = \frac{\partial}{\partial t} (x - t f(\theta))$$

$$= -f(\theta) - t f'(\theta) \theta_t$$

$$\Rightarrow \theta_t (1 + t f'(\theta)) = -f(\theta)$$

$$\theta_t = \frac{-f(\theta)}{1 + t f'(\theta)}$$

Clearly, θ_x and θ_t are undefined when

$$1 + t f'(\theta) = 0$$

$$\Rightarrow t = \frac{-1}{f'(\theta)}$$

The minimum time for the breaking time

$$t_b = -\min \left(-\frac{1}{f'(\theta)} \right) \neq$$

Shock wave: (Numerical) only:-

Solution: Ex-7:

Given

$$u_t + u u_x = 0 \quad \text{--- (i)}$$

$$\text{with } u(x, 0) = \begin{cases} 1 & \text{for } x \leq 0 \\ 0 & \text{for } x > 0 \end{cases} \quad \text{--- (ii)}$$

The characteristics equation of (i) is.

$$\frac{dx}{dt} = u, \quad \frac{du}{dt} = 0 \quad [\because u \text{ is constant}]$$

If $u=0$ then u is constant and if $u=1$ along the characteristics curve with slope 1.

$$\therefore \frac{dx}{dt} = 1 \text{ implies } x = t + x_0$$

Thus,

$$x(t) = \begin{cases} x_0 & \text{for } x > 0 \\ t + x_0 & \text{for } x \leq 0 \end{cases}$$

$$\text{since, } \phi_x = u \cdot u_x \quad \text{i.e. } d\phi_x = u \frac{du}{dx}$$

$$\text{integrating, } \phi_x = \frac{u^2}{2}$$

Now the shock speed is.

$$\begin{aligned} S = \frac{ds}{dt} &= \frac{\Delta \phi}{\Delta u} = \frac{\frac{1}{2}[(u^+)^2 - (u^-)^2]}{u^+ - u^-} \\ &= \frac{1}{2}(u^+ + u^-) \\ &= \frac{1}{2}(0 + 1) \\ &= \frac{1}{2} \end{aligned}$$

Here

Given.

$$2u_x + 2u_y - u = 0 \quad \text{--- (i)}$$

We first identify $a=2$, $b=2$ and $c=-u$.

The relation between the differential is

$$\frac{dx}{2} = \frac{dy}{2} = \frac{du}{-u} \quad \text{--- (ii)}$$

From eqn (ii) Take first and second term

$$\frac{dx}{2} = \frac{dy}{2}$$

integrating.

$$\frac{x}{2} = \frac{y}{2} + A$$

Again. take second and ~~third~~ ^{last} term

$$\frac{dy}{2} = \frac{du}{u}$$

integrating.

$$\frac{y}{2} = \ln(u) + B$$

$$u = e^{y/2B} = e^y \cdot C \quad \because C_1 = e^{y/2B}$$

since, $v = v(x, y)$ so. let

$$C_1 = F(A) = f\left(\frac{x-y}{2}\right)$$

Now the general solution is.

$$v = v(x, y) = f\left(\frac{x-y}{2}\right) e^{y/2}$$

Unit - 7:

Ex. Given

$$f(x) = f(x+p)$$

$$\cos\left(\frac{2\pi x}{k}\right) = \cos\left(\frac{2\pi}{k}(x+p)\right)$$

$$= \cos\left(\frac{2\pi x}{k}\right) \cdot \cos\left(\frac{2\pi p}{k}\right) - \sin\left(\frac{2\pi x}{k}\right) \cdot \sin\left(\frac{2\pi p}{k}\right)$$

Comparing and equating.

$$1 = \cos\left(\frac{2\pi p}{k}\right)$$

$$\cos 2\pi = \cos\left(\frac{2\pi p}{k}\right)$$

$$2\pi = \frac{2\pi p}{k}$$

$$p = k.$$

$$\therefore p = k \dots$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx.$$

$$= 0$$

Fourier Series with period 2π

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos n\pi$$

$$= \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot (-1)^n$$

$$= \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{2} - \frac{\pi^2}{6} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = 2 \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right)$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

proved.

Example 11:

Given

$$f(x) = |x| \quad \text{for } -\pi < x < \pi$$

clearly, $f(x)$ is 2π periodic function.
~~period of 2π~~

$$\text{Here, } f(-x) = |-x| \quad \text{for } -\pi < x < \pi$$
$$= |x| \quad \text{for } \pi > x > -\pi$$

$$= f(x)$$

This shows that the $f(x)$ is even function
So the $b_n = 0$.

The Fourier series is.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx) \quad \text{--- (i)}$$

Where,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx$$

$$= \frac{1}{2\pi} \cdot 2 \int_0^{\pi} x dx \quad [f(x) \text{ is even function}]$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$\text{and, } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx \quad [\because f(x) \text{ is even function}]$$

$$= \frac{2}{\pi} \left[x \frac{\sin nx}{n} - 1 \cdot \left(\frac{\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{\cos n0}{n^2} \right] \quad [\because \sin n\pi = 0]$$

$$= \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right]$$

Now. Put in eqⁿ (i)

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{\pi n^2} \frac{(-1)^n - 1}{2} \right] \cos nx \quad \text{--- (ii)}$$

This is the Fourier series of $f(x)$.

Now. We take $x=0$ then eqⁿ (ii) becomes

$$f(0) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{\pi n^2} \frac{(-1)^n - 1}{2} \right] \cos 0 \quad \text{--- (iii)}$$

$$0 = \frac{\pi}{2} + \sum_{n=\text{odd}} \frac{2}{\pi n^2} (-2) \cos 0$$

$$\therefore \frac{\pi}{2} = \frac{4}{\pi} \sum_{n=\text{odd}} \frac{1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n=\text{odd}} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example: - 14

Given

$$f(x) = \begin{cases} x & \text{for } -\pi/2 < x < \pi/2 \\ \pi - x & \text{for } \pi/2 < x < 3\pi/2 \end{cases}$$

Clearly, $f(x)$ is a periodic funcⁿ with
period = $\frac{3\pi}{2} - (-\pi/2) = \frac{3\pi}{2} + \frac{\pi}{2}$

$$= \frac{4\pi}{2}$$

$$= 2\pi.$$

replace by.

Here, $x = -x$

$$f(-x) = \begin{cases} -x & \text{for } -\pi/2 < -x < \pi/2 \\ \pi + x & \text{for } \pi/2 < -x < 3\pi/2 \end{cases}$$

$$= \begin{cases} -x & \text{for } -\pi/2 > x > \pi/2 \\ \pi + x & \text{for } \pi/2 > x > 3\pi/2 \end{cases}$$

$$\neq f(x)$$

then the given funcⁿ $f(x)$ is neither even nor odd.

Now. the Fourier series is given by

Date _____
Page _____

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (i)}$$

Where,

$$a_0 = \frac{1}{2\pi} \left[\int_{-\pi/2}^{\pi/2} x \, dx + \int_{\pi/2}^{3\pi/2} (\pi - x) \, dx \right]$$

$$= \frac{1}{2\pi} \left[\left[\frac{x^2}{2} \right]_{-\pi/2}^{\pi/2} + \pi \left[x \right]_{\pi/2}^{3\pi/2} - \left[\frac{x^2}{2} \right]_{\pi/2}^{3\pi/2} \right]$$

$$= \frac{1}{2\pi} \left[0 + \pi \left(\frac{9\pi}{2} - \frac{\pi}{2} \right) - \frac{1}{2} \left(\frac{9\pi^2}{4} - \frac{\pi^2}{4} \right) \right]$$

$$= \frac{1}{2\pi} \left[\pi^2 - \frac{1}{2} \frac{\pi^2}{4} \right]$$

$$= \frac{1}{2\pi} \left[\frac{8\pi^2}{8} - \frac{\pi^2}{16} \right]$$

$$= \frac{1}{2\pi} \left[\frac{8.5\pi^2}{8} - \frac{\pi^2}{16} \right]$$

(2)

Date _____
Page _____

Fourier series is $\frac{n\pi}{4} \cdot \frac{n\pi \cdot x}{4}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{4} + b_n \sin \frac{n\pi x}{4})$$

$$2l = + + = l = 1$$

$$\frac{1}{\pi} \int_{-1}^{1} f(x) \, dx = \frac{2}{\pi} \int_0^1 f(x) \, dx$$

$$l = \frac{\sin \frac{n\pi}{4} \cdot x}{4} \quad \cos \frac{n\pi \cdot x}{4}$$

Theorem 1:

Necessary Condⁿ:

Let $\vec{a}$ has a constant magnitude. then we have show that,

$$\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$$

We know,

$$\vec{a} \cdot \vec{a} = (|\vec{a}|)^2$$

Differentiating with respect to t ,

$$\frac{d(\vec{a} \cdot \vec{a})}{dt} = \frac{d(|\vec{a}|)^2}{dt}$$

$$\vec{a} \cdot \frac{d\vec{a}}{dt} + \vec{a} \cdot \frac{d\vec{a}}{dt} = \frac{d(|\vec{a}|)^2}{dt}$$

$$2\vec{a} \cdot \frac{d\vec{a}}{dt} = 0 \quad [\text{since } |\vec{a}| \text{ is constant}]$$

$$\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$$

Sufficient Condition:

Let $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$. then we have show that

$\vec{a}$ has a constant magnitude

We have.

$$\vec{a} \cdot \vec{a} = (|\vec{a}|)^2$$

Differentiating with respect to t ,

$$\frac{d(\vec{a} \cdot \vec{a})}{dt} = \frac{d(|\vec{a}|)^2}{dt}$$

$$2\left(\vec{a} \cdot \frac{d\vec{a}}{dt}\right) = 2 \cdot |\vec{a}| \frac{d|\vec{a}|}{dt}$$

$$0 = |\vec{a}| \frac{d|\vec{a}|}{dt} \quad \left[\because \vec{a} \cdot \frac{d\vec{a}}{dt} = 0 \right]$$

$$\therefore \vec{a} \neq 0 \text{ so, } |\vec{a}| \neq 0$$

$$\therefore |\vec{a}| \frac{d|\vec{a}|}{dt} = 0$$

$$\frac{d|\vec{a}|}{dt} = 0$$

This means $|\vec{a}|$ is constant. that is $\vec{a}$ has constant magnitude.

Theorem - 2:

Necessary Condition:

Let $\vec{a}$ has a constant direction. then we have to show that.

$$\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$$

We have

$$\vec{a} = a \hat{a}$$

$$\begin{aligned} \text{Here, } \vec{a} \times \frac{d\vec{a}}{dt} &= (a \hat{a}) \times \left(a \frac{d\hat{a}}{dt} + \frac{da}{dt} \hat{a} \right) \\ &= a \hat{a} \times \frac{da}{dt} \hat{a} \quad \because \frac{d\hat{a}}{dt} = \vec{0} \end{aligned}$$

$$\begin{aligned} &= \left(a \frac{da}{dt} \right) (\hat{a} \times \hat{a}) \\ &= a \frac{da}{dt} \cdot \vec{0} \quad [\because \hat{a} \times \hat{a} = \vec{0}] \\ &= \vec{0} \end{aligned}$$

$$\text{Thus, } \vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$$

Sufficient Condition

Let $\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$, then we have to show that $\vec{a}$ has a constant direction.

Here,

$$\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0} \Rightarrow (a \hat{a}) \times \frac{d(a \hat{a})}{dt} = \vec{0}$$

$$\Rightarrow a^2 (\hat{a} \times \frac{d\hat{a}}{dt}) = \vec{0}$$

$$\Rightarrow \hat{a} \times \frac{d\hat{a}}{dt} = \vec{0} \quad [\because a^2 \neq 0]$$

Again, $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$ that is,

$$\vec{a} \cdot \frac{d\vec{a}}{dt} = 0 \Rightarrow (a \hat{a}) \cdot \frac{d(a \hat{a})}{dt} = 0$$

$$\Rightarrow a^2 (\hat{a} \cdot \frac{d\hat{a}}{dt}) = 0$$

$$\Rightarrow \hat{a} \cdot \frac{d\hat{a}}{dt} = 0$$

$$\text{Thus, } \hat{a} \cdot \frac{d\hat{a}}{dt} = 0 \text{ and } \hat{a} \times \frac{d\hat{a}}{dt} = \vec{0}$$

This means $\hat{a}$ is constant. So $\vec{a}$ has a constant direction.

g. $y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$.

⇒ Here.

Given,

$$y'' + 4y' + 3y = e^{-t} \quad \text{--- (i)}$$

$$y(0) = y'(0) = 1 \quad \text{--- (ii)}$$

Taking L.T of eqⁿ (i) in both side,

$$L[y''] + 4L[y'] + 3L[y] = L(e^{-t})$$

$$s^2 L(y) - s(y(0)) - y'(0) + 4[sL(y) - y(0)]$$

$$+ 3L(y) = \frac{1}{s+1}$$

$$s^2 L(y) - s - 1 + 4sL(y) - 4 + 3L(y) = \frac{1}{s+1}$$

$$(s^2 + 4s + 3)L(y) - s - 5 = \frac{1}{s+1}$$

$$(s^2 + 4s + 3)L(y) = \frac{s+5}{s+1}$$

$$= \frac{s^2 + 6s + 5 + 1}{(s+1)}$$

$$(s^2 + 4s + 3)L(y) = \frac{s^2 + 6s + 6}{(s+1)(s+1)(s+3)}$$

$$L(y) = \frac{s^2 + 6s + 6}{(s+1)(s^2 + 4s + 3)} \quad \text{--- (iii)}$$

Let,

$$\frac{s^2 + 6s + 6}{(s+1)(s^2 + 4s + 3)} = \frac{A}{(s+1)} + \frac{Bs + C}{(s^2 + 4s + 3)}$$

$$s^2 + 6s + 6 = As^2 + 4As + 3A + Bs^2 + Cs + Bs + C$$

$$s^2 + 6s + 6 = (A+B)s^2 + (4A+B+C)s + (3A+C)$$

Equating,

$$A+B=1$$

$$\therefore A = -B \quad \text{--- (a)}$$

$$3A+C=6$$

$$-3B+6+C=6$$

$$4A+B+C=6$$

$$-4B+B+C=6$$

$$-3B+C=6$$

$$\therefore C = 6 + 3B \quad \text{--- (b)}$$

Bessel's Equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$$

is called Bessel's Equation of order n .

The Bessel's Equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0 \quad \text{--- (i)}$$

Suppose the series.

$$y = \sum_{m=0}^{\infty} a_m x^{m+k} \quad \text{--- (ii)}$$

then,

$$y' = \sum_{m=0}^{\infty} (m+k) a_m x^{m+k-1} \quad \text{and}$$

$$y'' = \sum_{m=0}^{\infty} (m+k)(m+k-1) a_m x^{m+k-2}$$

Substituting y in eqn (i) we get

$$\Rightarrow x^2 \sum_{m=0}^{\infty} (m+k)(m+k-1) a_m x^{m+k-2} + x \sum_{m=0}^{\infty} (m+k) a_m x^{m+k-1} + (x^2 - n^2) \sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+k)(m+k-1) a_m x^{m+k} + \sum_{m=0}^{\infty} (m+k) a_m x^{m+k} + (x^2 - n^2) \sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$+ (x^2 - n^2) \sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\sum_{m=0}^{\infty} \left[(m+k)(m+k-1) + (m+k) + (x^2 - n^2) \right] a_m x^{m+k} = 0$$

$$= \sum_{m=0}^{\infty} \left[(m+k)^2 - n^2 \right] a_m x^{m+k} + \sum_{m=0}^{\infty} a_m x^{m+k+2} = 0$$

--- (iii)

The lowest degree in (iii) is x^k corresponding to $m=0$, equating

$$(k^2 - n^2) a_0 = 0 \Rightarrow k^2 = n^2 \text{ as } a_0 \neq 0$$

$$k = \pm n$$

Equating x^{k+1} to zero.

$$[(k+1)^2 - n^2] a_1 = 0$$

$a_1 = 0$ as $(k+1)^2 - n^2 \neq 0$ for $k \neq \pm n$

Again, Equating x^{m+k-2} we get,

$$[(m+k+2)^2 - n^2] a_{k+2} + a_k = 0$$

$$\Rightarrow a_{k+2} = -\frac{a_k}{(m+k+2)^2}$$

Putting $m = 1, 3, 5 \dots$ we get,

$$a_3 = a_5 = a_7 = \dots = 0$$

Putting $m = 0, 2, 4 \dots$ we get,

$$a_2 = -\frac{a_0}{(k+2)^2 - n^2}$$

$$a_4 = \frac{a_2}{(k+4)^2 - n^2} = \frac{a_0}{[(k+2)^2 - n^2][(k+4)^2 - n^2]}$$

The solution of (i) is,

$$y = a_0 x^k \left[1 - \frac{1}{(k+2)^2 - n^2} x^2 + \frac{1}{[(k+2)^2 - n^2][(k+4)^2 - n^2]} x^4 + \dots \right]$$

for $k = n$,

$$y_1 = a_0 x^n$$

Bessel's funcⁿ:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0.$$

properties

①: $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(2) $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$

properties:

① $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

② $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

③ $2n J_n(x) = x J_{n-1}(x) + x J_{n+1}(x)$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n+k)!} \left(\frac{x}{2}\right)^{2k+n}$$

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+1/2)!} \left(\frac{x}{2}\right)^{2k+1/2}$$

$$= \frac{(-1)^0}{0! (\frac{1}{2})!} \left(\frac{x}{2}\right)^{1/2} + \frac{(-1)^1}{1! (1+1/2)!} \left(\frac{x}{2}\right)^{3/2} +$$

$$\frac{(-1)^2}{2! (2+1/2)!} \left(\frac{x}{2}\right)^{5/2} + \dots$$

$$= \frac{1}{\frac{1}{2} \sqrt{\pi/2}} \left(\frac{x}{2}\right)^{1/2} - \frac{1}{\frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi/2}} \left(\frac{x}{2}\right)^{3/2} +$$

$$+ \frac{1}{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi/2}} \left(\frac{x}{2}\right)^{5/2} + \dots$$

$$= \left(\frac{x}{2}\right)^{1/2} \left[\frac{2}{\sqrt{\pi}} - \frac{2x^2}{3 \times 2 \sqrt{\pi}} + \frac{2x^4}{5 \times 4 \times 3 \times 2 \sqrt{\pi}} - \dots \right]$$

$$2 \left(\frac{x}{2\pi} \right)^{1/2} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots \right]$$

$$= \sqrt{\frac{2}{2\pi}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

$$\vec{F} = \nabla \phi$$

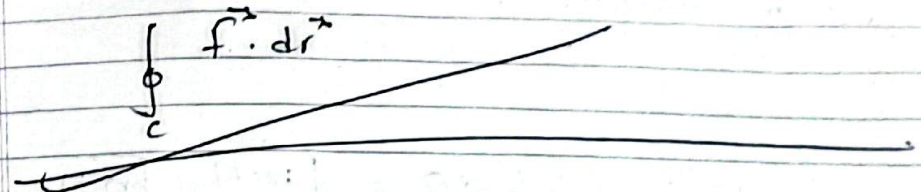
$$\nabla \phi \cdot d\vec{r} = \left(\quad \right) \cdot dx \hat{i} +$$

$$= d\phi$$

$$\vec{F} \cdot d\vec{r} = d\phi$$

$$d\phi = \left(\quad \right) \cdot \left(\quad \right)$$

line integral

$$\int_C \vec{F} \cdot d\vec{r}$$


surface integral

$$\iint \vec{F} \cdot \vec{n} \, d\sigma$$

$$\vec{r}_u \times \vec{r}_v = \vec{N} = \vec{n}$$

$$\vec{F} \cdot \vec{n} = \left(\quad \right)$$

$$\iint$$